Periodically wrinkled plate model of the Föppl-von Kármán type

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Abstract In this paper we derive, by means of Γ-convergence, the periodically wrinkled plate model starting from three dimensional nonlinear elasticity. We assume that the thickness of the plate is h^2 and that the mid-surface of the plate is given by $(x_1, x_2) \to (x_1, x_2, h^2\theta(\frac{x_1}{h}, \frac{x_2}{h}))$, where θ is $[0,1]^2$ periodic function. We also assume that the strain energy of the plate has the order $h^8 = (h^2)^4$, which corresponds to the Föppl-von Kármán model in the case of the ordinary plate. The obtained model mixes the bending part of the energy with the stretching part.

1 Introduction

The study of thin structures is the subject of numerous works in the theory of elasticity. Many authors have proposed two-dimensional shell and plate models and we come to the problem of their justification. There is a vast literature on the subject of plates and shells (see [7,8]).

The justification of the model of plates and shells, by using Γ-convergence is well established. The first works in that direction are [15,16]. The thickness of the plate is assumed to be h, a small parameter, and the external loads are assumed to be of the order 0. The obtained model for plate and shells differs

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from the one obtained by the formal asymptotic expansion in the sense that additional relaxation of the energy functional is done.

From the pioneering work of Friesecke, James, Müller [11] higher order models of plates and shells are justified from three dimensional nonlinear elasticity (see [11,12,13,14,18,19] . Here, higher order, relates that we assume that the magnitude of the external loads (i.e. of the strain energy) behaves like $h^\alpha, \alpha > 0$ (i.e. $h^\beta, \beta > 0$). Depending on different parameter α different lower-dimensional models are obtained (see [12]).

Different influence of the imperfections of the domain on the model is also discussed in the literature. In [5] it is assumed that the stored energy function oscillates with the order h, as the thickness of the plate, but the strain energy (after divided by the order of volume h) is assumed to be of the order 0. This model thus corresponds to the one given in [15] for the ordinary plate. Also, the influence of the specific type of the imperfections of the domain on the Föppl-von Kármán plate model is discussed in [20]. The special case of shallow shell and weakly curved rod is discussed in [24,25]. All these models do not include periodic wrinkles which we discuss here. Here we assume that the thickness of the plate is h^2 and that the mid-surface of the plate is given by $(x_1, x_2) \to (x_1, x_2, h^2\theta(\frac{x_1}{h}, \frac{x_2}{h}))$, where θ is $[0,1]^2$ periodic function. We also assume that the strain energy of the plate (divided by the order of volume h^2) has the order $h^8 = (h^2)^4$, which corresponds to the Föppl-von Kármán model in the case of the ordinary plate. The obtained model mixes the bending part of the energy with the stretching part.

It could be interesting if we could generalize these periodic wrinkles to the oscillations of the stored energy function (like it is done in [5]). But it is also important to see that our model depends on the pre-deformation θ_0 (see (4.60)), thus not only on the derivatives of θ. To deal with periodic wrinkles we use the tool of two-scale convergence. The wrinkled plates model, derived from two dimensional linear Koiter shell model, are derived in [3,2]. This model (and its linearization) is different from those ones, which is expected, since we derive the model from three dimensional nonlinear theory and thickness of the plate is of the same order as the amplitude of the mid-surface.

Throughout the paper $\bar{A}$ or $\{A\}^-$ denotes the closure of the set. By a domain we call a bounded open set with Lipschitz boundary. $\mathbf{I}$ denotes the identity matrix, by SO(3) we denote the rotations in $\mathbb{R}^3$ and by so(3) the set of antisymmetric matrices 3×3. By $\mathbb{R}^{n \times n}_{\mathrm{sym}}$ we denote the set of symmetric matrices of the dimension $n \times n$. x' stands for (x_1, x_2). $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the vectors of the canonical base in $\mathbb{R}^3$. By id we denote the identity mapping $id(x) = x$. $\to$ denotes the strong convergence and $\rightharpoonup$ the weak convergence. By $\mathbf{A} \cdot \mathbf{B}$ we denote $\mathrm{tr}(\mathbf{A}^T\mathbf{B})$. We suppose that the Greek indices α, β take the values in the set $\{1, 2\}$ while the Latin indices i, j take the values in the set $\{1, 2, 3\}$.

2 Setting up the problem

Let ω be a two-dimensional domain with Lipschitz boundary in the plane spanned by $\boldsymbol{e}_1, \boldsymbol{e}_2$; the generic point in ω we denote by $x' = (x_1, x_2)$. The canonical cell in $\mathbb{R}^2$ we denote by $Y = [0, 1]^2$; the generic point in Y is $y = (y_1, y_2)$. By a periodically wrinkled plate we mean a shell defined in the following way. Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a Y-periodic function of class C^2. We call θ the shape function. We consider a three-dimensional elastic shell occupying in its reference configuration the set $\{\hat{\Omega}^h\}^-$, where

$$\hat{\Omega}^h = \boldsymbol{\Theta}^h(\Omega^h), \ \Omega^h = \omega \times (-\frac{h^2}{2}, \frac{h^2}{2});$$

the mapping $\boldsymbol{\Theta}^h : \{\hat{\Omega}^h\}^- \to \mathbb{R}^3$ is given by

$$\boldsymbol{\Theta}^h(x^h) = (x_1, x_2, h^2\theta(\frac{x_1}{h}, \frac{x_2}{h})) + x_3^h \boldsymbol{n}^h(x_1, x_2)$$

for all $x^h = (x', x_3^h) \in \bar{\Omega}^h$, where $\boldsymbol{n}^h$ is a unit normal vector to the middle surface $\boldsymbol{\Theta}^h(\overline{\omega})$ of the shell. By Ω we denote Ω^1 and by x_3 we denote $\frac{x_3^h}{h^2}$. At each point of the surface $\bar{\omega}$ the vector $\boldsymbol{n}^h$ is given by

$$\boldsymbol{n}^h(x_1, x_2) = (n^h(x_1, x_2))^{-1/2}(-h\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h}), -h\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h}), 1),$$

where

$$n^h(x_1, x_2) = h^2\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + h^2\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + 1.$$

By inverse function theorem it can be easily seen that for $h \leq h_0$ small enough $\boldsymbol{\Theta}^h$ is a C^1 diffeomorphism (the global injectivity can be proved by adapted compactness argument, see [8, Thm 3.1-1] for the ordinary shell). Let us by $\theta_0 : \mathbb{R}^2 \to \mathbb{R}$ denote the function:

$$\theta_0 = \theta - \langle\theta\rangle, \quad \langle\theta\rangle := \int_Y \theta dy. \tag{2.1}$$

The following theorem is easy to prove.

Theorem 1 *Then there exists $h_0 = h_0(\theta) > 0$ such that the Jacobian matrix $\nabla\boldsymbol{\Theta}^h(x^h)$ is invertible for all $x^h \in \bar{\Omega}^h$ and all $h \leq h_0$. Also there exists $C > 0$ such that for $h \leq h_0$ we have*

$$\det \nabla\boldsymbol{\Theta}^h = 1 + h^2\delta^h(x^h), \tag{2.2}$$

and

$$\nabla\boldsymbol{\Theta}^h(x^h) = \mathbf{I} - h\mathbf{C}(x_1, x_2) - h^2\mathbf{D}(x_1, x_2, x_3) + h^3\mathbf{R}_1^h(x^h), \tag{2.3}$$

$$(\nabla\boldsymbol{\Theta}^h(x^h))^{-1} = \mathbf{I} + h\mathbf{C}(x_1, x_2) + h^2\mathbf{E}(x_1, x_2, x_3) + h^3\mathbf{R}_2^h(x^h), \tag{2.4}$$

$$\|(\nabla\boldsymbol{\Theta}^h) - \mathbf{I}\|_{L^\infty(\Omega^h;\mathbb{R}^{3\times3})}, \|(\nabla\boldsymbol{\Theta}^h)^{-1} - \mathbf{I}\|_{L^\infty(\Omega^h;\mathbb{R}^{3\times3})} < Ch, \tag{2.5}$$

where

$$\mathbf{C}(x_1, x_2) = \begin{pmatrix} 0 & 0 & \partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h}) \\ 0 & 0 & \partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h}) \\ -\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h}) & -\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h}) & 0 \end{pmatrix}, \qquad (2.6)$$

$$\mathbf{D}(x_1, x_2, x_3) = $$
$$\begin{pmatrix} x_3\partial_{11}\theta(\frac{x_1}{h}, \frac{x_2}{h}) & x_3\partial_{12}\theta(\frac{x_1}{h}, \frac{x_2}{h}) & 0 \\ x_3\partial_{12}\theta(\frac{x_1}{h}, \frac{x_2}{h}) & x_3\partial_{22}\theta(\frac{x_1}{h}, \frac{x_2}{h}) & 0 \\ 0 & 0 & \frac{1}{2}\Big(\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + \partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})^2\Big) \end{pmatrix},$$
$$(2.7)$$

$$\mathbf{E}(x_1, x_2, x_3) = \mathbf{C}^2(x_1, x_2) + \mathbf{D}(x_1, x_2, x_3)$$
$$= (\mathbf{E}_1(x_1, x_2, x_3), \mathbf{E}_2(x_1, x_2, x_3), \mathbf{E}_3(x_1, x_2, x_3)), \quad (2.8)$$

$$\mathbf{E}_1(x_1, x_2, x_3) = \begin{pmatrix} -\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + x_3\partial_{11}\theta(\frac{x_1}{h}, \frac{x_2}{h}) \\ -\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h}) + x_3\partial_{12}\theta(\frac{x_1}{h}, \frac{x_2}{h}) \\ 0 \end{pmatrix}, \quad (2.9)$$

$$\mathbf{E}_2(x_1, x_2, x_3) = \begin{pmatrix} -\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h}) + x_3\partial_{12}\theta(\frac{x_1}{h}, \frac{x_2}{h}) \\ -\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + x_3\partial_{22}\theta(\frac{x_1}{h}, \frac{x_2}{h}) \\ 0 \end{pmatrix}, \quad (2.10)$$

$$\mathbf{E}_3(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\Big(\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + \partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})^2\Big) \end{pmatrix}, \qquad (2.11)$$

and $\delta^h : \bar{\Omega}^h \to \mathbb{R}$, $\mathbf{R}_k^h : \bar{\Omega}^h \to \mathbb{R}^{3\times 3}$, $k = 1, 2$ are functions which satisfy

$$\sup_{0 < h \le h_0} \max_{x^h \in \bar{\Omega}^h} |\delta^h(x^h)| \le C_0, \ \sup_{0 < h \le h_0} \max_{i,j} \max_{x^h \in \bar{\Omega}^h} |\mathbf{R}_{k,ij}^h(x^h)| \le C_0, \ k = 1, 2,$$

for some constant $C_0 > 0$.

Proof. It is easy to see

$$\boldsymbol{n}^h(x_1, x_2) = \boldsymbol{e}_3 - h\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})\boldsymbol{e}_1 - h\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})\boldsymbol{e}_2$$
$$- \frac{h^2}{2}\Big(\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})^2 + \partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})^2\Big)\boldsymbol{e}_3 + h^3\mathbf{o}_1^h(x_1, x_2), \ (2.12)$$

where

$$\sup_{0 < h \le h_0} \max_{x^h \in \bar{\Omega}^h} |\mathbf{o}_1^h(x_1, x_2)| \le C, \ \sup_{0 < h \le h_0} \max_{x^h \in \bar{\Omega}^h} |\partial_\alpha \mathbf{o}_1^h(x_1, x_2)| \le \frac{C}{h},$$

for some $C > 0$ and $\alpha = 1, 2$. From the definition $\boldsymbol{\Theta}^h$ we conclude that

$$\nabla \boldsymbol{\Theta}^h(x', x_3^h) = \mathbf{I} + h\partial_1\theta(\frac{x_1}{h}, \frac{x_2}{h})\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + h\partial_2\theta(\frac{x_1}{h}, \frac{x_2}{h})\boldsymbol{e}_3 \otimes \boldsymbol{e}_2$$
$$+ (x_3^h\partial_1\boldsymbol{n}^h(\frac{x_1}{h}, \frac{x_2}{h}), x_3^h\partial_2\boldsymbol{n}^h(\frac{x_1}{h}, \frac{x_2}{h}), \boldsymbol{n}^h). \tag{2.13}$$

The relation (2.3) is the direct consequence of the relations (2.12) and (2.13). The relations (2.2), (2.4), (2.5) are the direct consequences of the relation (2.3).

The starting point of our analysis is the minimization problem for the wrinkled plate. The strain energy of the wrinkled plate is given by

$$K^h(\boldsymbol{y}) = \int_{\hat{\Omega}^h} W(\nabla\boldsymbol{y}(x))dx,$$

where $W : \mathbb{M}^{3\times3} \to [0, +\infty]$ is the stored energy density function. W is Borel measurable and, as in [11, 12, 13], is supposed to satisfy

  i) W is of class C^2 in a neighborhood of SO(3);
  ii) W is frame-indifferent, i.e., $W(\mathbf{F}) = W(\mathbf{RF})$ for every $\mathbf{F} \in \mathbb{R}^{3\times3}$ and $\mathbf{R} \in \text{SO}(3)$;
  iii) $W(\mathbf{F}) \geq C_W \text{dist}^2(\mathbf{F}, \text{SO}(3))$, for some $C_W > 0$ and all $\mathbf{F} \in \mathbb{R}^{3\times3}$, $W(\mathbf{F}) = 0$ if $\mathbf{F} \in \text{SO}(3)$.

By $Q_3 : \mathbb{R}^{3\times3} \to \mathbb{R}$ we denote the quadratic form $Q_3(\mathbf{F}) = D^2W(\mathbf{I})(\mathbf{F}, \mathbf{F})$ and by $Q_2 : \mathbb{R}^{2\times2} \to \mathbb{R}$ the quadratic form,

$$Q_2(\mathbf{G}) = \min_{\boldsymbol{a}\in\mathbb{R}^3} Q_3(\mathbf{G} + \boldsymbol{a} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{a}), \tag{2.14}$$

obtained by minimizing over the stretches in the x_3 directions. Using ii) and iii) we conclude that both forms are positive semi-definite (and hence convex), equal to zero on antisymmetric matrices and depend only on the symmetric part of the variable matrix, i.e. we have

$$Q_3(\mathbf{G}) = Q_3(\text{sym}\,\mathbf{G}), \quad Q_2(\mathbf{G}) = Q_2(\text{sym}\,\mathbf{G}). \tag{2.15}$$

Also, from ii) and iii), we can conclude that both forms are positive definite (and hence strictly convex) on symmetric matrices. For the special case of isotropic elasticity we have

$$Q_3(\mathbf{F}) = 2\mu|\frac{\mathbf{F} + \mathbf{F}^T}{2}|^2 + \lambda(\text{tr}\,\mathbf{F})^2,$$
$$Q_2(\mathbf{G}) = 2\mu|\frac{\mathbf{G} + \mathbf{G}^T}{2}|^2 + \frac{2\mu\lambda}{2\mu + \lambda}(\text{tr}\,\mathbf{G})^2. \tag{2.16}$$

Since the strain energy is the most difficult part to deal with (see Remark.....) we shall look for the Γ-limit of the functional

$$I^h(\boldsymbol{y}) = \frac{1}{h^8}\frac{1}{h^2}K^h(\boldsymbol{y}).$$

The reason why we divide by h^8 is because we are interested in the Föppl-von Kármán type model of the wrinkled plate (the thickness of the plate is h^2). The reason why we additionally divide K^h by h^2 is because the volume of $\hat{\Omega}^h$ is decreasing with the order h^2. In the third section we prove some technical results about two scale convergence which we need later, in the forth section we prove the Γ-convergence result. To prove it we firstly need the compactness result which tells us how the displacements of the energy order h^8 look like and secondly we have to prove lower and upper bound which is standard in Γ-analysis.

3 Two-scale convergence

For the notion of two scale convergence see [4, 21]. Here $\omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $Y = [0, 1]^n$. For $k = (k_i)_{i=1,\dots,n} \in \mathbb{Z}^n$ we denote by $|k| = (\sum_{i=1}^n k_i^2)^{1/2}$.

We denote by $C_{\#}^k(Y)$ the space of k-differentiable functions with continuous k-th derivative in $\mathbb{R}^2$ which are periodic of period Y. Then $L_{\#}^2(Y)$ (respectively $H_{\#}^m(Y)$ is the completion for the norm of $L^2(Y)$ (respectively $H^m(Y)$) of $C_{\#}^\infty(Y)$. Remark that $L_{\#}^2(Y)$ actually coincides with the space of functions in $L^2(Y)$ extended by Y-periodicity to the whole of $\mathbb{R}^n$. $H_{\#}^1(Y)$ coincides with the space of functions in $H^1(Y)$ which are Y-periodic at the boundary in the sense of traces and $H_{\#}^2(Y)$ coincides with the space of functions in $H^2(Y)$ which are with their first derivatives Y-periodic at the boundary in the sense of traces etc. Using the Fourier transform on torus it can be seen that (see e.g. [23])

$$H_{\#}^m(Y) = \{u, \ u(y) = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i k \cdot y}, \overline{c}_k = c_{-k}, \sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k|^2 < \infty\}, \quad (3.17)$$

and the norm $\|u\|_{H_{\#}^m(Y)}$ is equivalent to the norm $\{\sum_{k \in \mathbb{Z}^2} (1 + |k|^{2m}) |c_k|^2\}^{1/2}$. We also set

$$\dot{H}_{\#}^m(Y) = \{u \text{ of type } (3.17), \ c_0 = 0\} = \{u, u \in H_{\#}^m(Y), \ \int_Y u = 0\}.$$

The restricted norm $\|u\|_{\dot{H}_{\#}^m(Y)}$ is equivalent to the norm $\{\sum_{k \in \mathbb{Z}^2} |k|^{2m} |c_k|^2\}^{1/2}$. The space $L^2(\omega; C_{\#}(Y))$ denotes the space of measurable and square integrable in $x \in \omega$ with values in the Banach space of continuous functions, Y-periodic in y. In the analogous way one can define $L^2(\omega; H_{\#}^m(Y))$. Using the representation (3.17) it can be seen that

$$L^2(\omega; H_{\#}^m(Y)) = \{u, \ u(x, y) = \sum_{k \in \mathbb{Z}^n} c_k(x) e^{2\pi i k \cdot y}, c_k \in L^2(\omega; \mathbb{R}^2), \overline{c}_k = c_{-k},$$
$$\sum_{k \in \mathbb{Z}^n} |k|^{2m} \|c_k\|_{L^2(\omega)}^2 < \infty\}, \quad (3.18)$$

where we have by $\overline{c}_k$ denoted the conjugate of c_k. The norm $\|u\|_{L^2(\omega;H^m_{\#}(Y))}$ is equivalent to the norm

$$\{\sum_{k\in\mathbb{Z}^2}(1+|k|^{2m})\|c_k\|^2_{L^2(\omega)}\}^{1/2}.$$

The space $\mathcal{D}(\omega;C^\infty_{\#}(Y))$ denotes the space of infinitely differentiable functions which take values in $C^\infty_{\#}(Y)$ with compact support in ω. It is easily seen that this space is dense in $L^2(\omega;H^m_{\#}(Y))$. In fact it can be seen that the space of finite linear combinations

$$FL(\omega;C^\infty_{\#}(Y)) = \{u, \exists n \in \mathbf{N}\ u(x,y) = \sum_{k\in\mathbb{Z}^n,\ |k|\leq n} c_k(x)e^{2\pi ik\cdot y},$$

$$c_k \in C^\infty_0(\omega;\mathbb{R}^2), \overline{c}_k = c_{-k}\}$$

is dense in $L^2(\omega;H^m_{\#}(Y))$.

Definition 1 A sequence $(u_h)_{h>0}$ of functions in $L^2(\omega)$ converges two-scale to a function u_0 belonging to $L^2(\omega \times Y)$ if for every $\psi \in L^2(\omega;C_{\#}(Y))$,

$$\int_\omega u_h(x)\psi(x,\frac{x}{h}) \to \int_\omega\int_Y u_0(x,y)\psi(x,y) \quad \text{as } h \to 0.$$

By $\rightharpoonup\rightharpoonup$ we denote the two-scale convergence. The following theorems are given in [4].

Theorem 2 *Let $f \in L^1(\omega;C_{\#}(Y))$. Then $f(x,\frac{x}{h})$ is a measurable function on ω for which it is valid:*

$$\|f(x,\frac{x}{h})\|_{L^1(\omega)} \leq \int_\omega \sup_{y\in Y}|f(x,y)|dy =: \|f\|_{L^1(\omega;C_{\#}(Y))}, \tag{3.19}$$

and

$$\lim_{h\to 0}\int_\omega f(x,\frac{x}{h})dx = \int_\omega\int_Y f(x,y)dydx. \tag{3.20}$$

Theorem 3 *From each bounded sequence $(u_h)_{h>0}$ in $L^2(\omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\omega \times Y)$ such that this subsequence two-scale converges to u_0.*

The following theorem tells us about the form of oscillations of order h of weakly convergent sequences in $H^1(\omega)$.

Theorem 4 *Let (u_h) be a bounded sequence in $H^1(\omega)$ which converges weakly to a limit $u \in H^1(\omega)$. Then u_h two-scale converges to $u(x)$, and there exists a unique function $u_1(x,y) \in L^2(\omega;\dot{H}^1_{\#}(Y))$ such that, up to a subsequence, $(\nabla u_h)_{h>0}$ two-scale converges to $\nabla_x u + \nabla_y u_1(x,y)$.*

Remark 1 In the definition of two-scale convergence we have taken the test functions to be in the space $L^2(\omega;C_{\#}(Y))$. When we are dealing with the sequence of the functions which are bounded in $L^2(\omega)$ it is enough to take the test function to be in the space $\mathcal{D}(\omega;C^\infty_{\#}(Y))$.

The following lemmas will be needed later.

Lemma 1 *Let $(u_h)_{h>0}$ be a bounded sequence in $W^{1,2}(\omega)$ and let there exists a constant $C > 0$ such that $\|u_h\|_{L^2(\omega)} \leq Ch^2$. Then we have that $(u_h)_{h>0}$ and $(\nabla u_h)_{h>0}$ two-scale converge to 0.*

Proof. That $(u_h)_{h>0}$ two-scale converges to zero is the direct consequence of the fact that strong convergence implies two-scale convergence to the same limit (not depending on $y \in Y$). Let us now take $\psi \in \mathcal{D}(\omega; C_{\#}^{\infty}(Y))$. Then we have

$$\int_{\omega} \partial_i u_h(x)\psi(x, \frac{x}{h})dx = -\int_{\omega} u_h(x)\partial_x \psi(x, \frac{x}{h})dx - \frac{1}{h}\int_{\omega} u_h(x)\partial_y \psi(x, \frac{x}{h})dx \tag{3.21}$$

Since the both terms in (3.21) converge to 0, due to the fact that $\|u_h\|_{L^2(\omega)} \leq Ch^2$ we have the claim.

The following characterization of the potentials is needed

Lemma 2 *Let $\boldsymbol{u} \in L^2(\omega \times Y; \mathbb{R}^n)$ be such that for each $\boldsymbol{\psi} \in \mathcal{D}(\omega; C_{\#}^{\infty}(Y))^n$ which satisfies $\mathrm{div}_y \boldsymbol{\psi} = 0$, $\forall x, y$ we have that*

$$\int_{\Omega} \int_Y \boldsymbol{u}(x, y)\boldsymbol{\psi}(x, y)dydx = 0.$$

Then there exists a unique function $v \in L^2(\omega; \dot{H}_{\#}^1(Y))$ such that $\nabla_y v = \boldsymbol{u}$. In the same way, let $\boldsymbol{U} \in L^2(\omega \times Y; \mathbb{R}^{n \times n})$ be such that for each $\boldsymbol{\Psi} \in \mathcal{D}(\omega; C_{\#}^{\infty}(Y))^{n \times n}$ which satisfies $\sum_{i,j=1}^{n} \partial_{y_i y_j} \boldsymbol{\Psi}^{ij} = 0$, $\forall x, y$ we have that

$$\int_{\Omega} \int_Y \boldsymbol{U}(x, y) \cdot \boldsymbol{\Psi}(x, y)dydx = 0. \tag{3.22}$$

Then there exists a unique function $v \in L^2(\omega; \dot{H}_{\#}^1(Y))$ such that $\nabla_y^2 v = \boldsymbol{U}$.

Proof. We shall only prove the second claim since the first goes in the analogous way. Let us define the operator $\nabla_y^2 : L^2(\omega; \dot{H}_{\#}^2(Y)) \to L^2(\omega \times Y; \mathbb{R}^{n \times n})$ by $v \to \nabla_y^2 v$. Let us identify the space $L^2(\omega; \dot{H}_{\#}^2(Y))$ with the sequences of functions

$$L^2(\omega; \dot{H}_{\#}^2(Y)) = \{(c_k)_{k \in \mathbb{Z}^n}, c_k \in L^2(\omega; \mathbb{R}^2), \overline{c}_k = c_{-k},$$
$$\int_{\omega} \sum_{k \in \mathbb{Z}^n} |k|^4 |c_k(x)|^2 dx < \infty\},$$

with the norm

$$\|(c_k)_k\|^2 = \int_{\omega} \sum_{k \in \mathbb{Z}^n} |k|^4 |c_k(x)|^2 dx$$

and the space $L^2(\omega \times Y; \mathbb{R}^{n \times n})$ with

$$L^2(\omega \times Y; \mathbb{R}^{n \times n}) = \{(c_k^{ij})_{k \in \mathbb{Z}^n, i,j=1,\ldots,n}, c_k \in L^2(\omega; \mathbb{R}^2), \overline{c}_k^{ij} = c_{-k}^{ij},$$

$$\int_\omega \sum_{k \in \mathbb{Z}^n} \sum_{i,j=1,\ldots n} |c_k^{ij}(x)|^2 dx < \infty\},$$

with the norm

$$\|(c_k^{ij})_k\|^2 = \int_\omega \sum_{k \in \mathbb{Z}^n} \sum_{i,j=1,\ldots,n} |c_k^{ij}(x)|^2 dx$$

The operator ∇_y^2 operates in the following way

$$\nabla_y^2 (c_k)_k = ((k_i k_j c_k)_{i,j=1,\ldots,n})_{k \in \mathbb{Z}^n}.$$

It is easily seen that ∇_y^2 is continuous and one to one. We shall prove that it is enough to demand the condition (3.22) for $\boldsymbol{\Psi} \in FL(\omega; C_\#^\infty(Y))^{n \times n}$. Using the properties of the Fourier transform the condition (3.22) can be interpreted in the following way: For given $((\boldsymbol{U}_k^{ij})_{i,j=1,\ldots,n})_{k \in \mathbb{Z}^n} \in L^2(\omega \times Y; (\mathbb{R}^2)^{n \times n})$ and every $((d_k^{ij})_{i,j=1,\ldots,n})_{k \in \mathbb{Z}^2} \in FL(\omega; C_\#^\infty(Y))^{n \times n}$ which satisfies the property

$$\sum_{i,j=1,\ldots,n} k_i k_j d_k^{ij} = 0, \ \forall k \in \mathbb{Z}^n, \tag{3.23}$$

we have that

$$\int_\omega \sum_{k \in \mathbb{Z}^2} \sum_{i,j=1,\ldots n} \boldsymbol{U}_k^{ij}(x) \overline{d}_k^{ij}(x) dx = 0.$$

By fixing $k^0 \in \mathbb{Z}^n$ and taking $d_{k^0}^{ij} \in C_0^\infty(\omega; \mathbb{R}^2)$, which satisfies

$$\sum_{i,j=1,\ldots,n} k_i^0 k_j^0 d_{k^0}^{ij} = 0$$

and defining $d_{-k^0}^{ij} = \overline{d}_{k^0}^{ij}$, $d_k^{ij} = 0$, $\forall k \neq k^0, -k^0$, $i,j = 1,\ldots,n$ we conclude that

$$\int_\omega \operatorname{Re}(\boldsymbol{U}_{k^0}^{ij}(x) \overline{d}_{k^0}^{ij}(x)) dx = 0.$$

From this it can be easily seen that there exists $v_{k^0} \in L^2(\omega; \mathbb{R}^2)$ such that $\boldsymbol{U}_{k^0}^{ij} = k_i^0 k_j^0 v_{k^0}$, for all $i,j = 1,\ldots,n$. This is valid for an arbitrary $k_0 \in \mathbb{Z}^n$ and we can easily conclude from the fact $((\boldsymbol{U}_k^{ij})_{i,j=1,\ldots,n})_{k \in \mathbb{Z}^2} \in L^2(\omega \times Y; (\mathbb{R}^2)^{n \times n})$ that $\int_\omega \sum_{k \in \mathbb{Z}^n} |k|^4 |v_k(x)|^2 dx < \infty$. Now we have the claim by taking $v(x,y) = (v_k(x))_k \equiv \sum_{k \in \mathbb{Z}^n} v_k(x) e^{2\pi i k \cdot y}$.

Lemma 3 *Let $(u_h)_{h>0}$ be a sequence which converges strongly to u in $W^{1,2}(\omega)$. Let $(\boldsymbol{v}_h)_{h>0}$ be a sequence which is bounded in $W^{1,2}(\omega; \mathbb{R}^n)$ and for which is valid*

$$\|\nabla u_h - \boldsymbol{v}_h\|_{L^2(\omega; \mathbb{R}^n)} \leq Ch^2, \tag{3.24}$$

for some $C > 0$. Then there exists a unique $v \in L^2(\omega; \dot{H}_\#^2(Y))$ such that $\nabla \boldsymbol{v}_h \rightharpoonup\rightharpoonup \nabla^2 u(x) + \nabla_y^2 v(x,y)$.

Proof. It is easily seen that $\boldsymbol{v}_h \to \nabla u$ weakly in $W^{1,2}(\omega; \mathbb{R}^n)$ and thus $u \in W^{2,2}(\omega)$. By using Theorem 3 we conclude that there exists $\boldsymbol{\Phi}^{ij} \in L^2(\omega \times Y)$ such that

$$\nabla \boldsymbol{v}_h^i \rightharpoonup\rightharpoonup (\partial_{i1} u(x) + \boldsymbol{\Phi}^{i1}(x,y), \ldots, \partial_{in} u(x) + \boldsymbol{\Phi}^{in}(x,y)).$$

To show the existence of v we shall use lemma 2. Let us take $\boldsymbol{\Psi} \in \mathcal{D}(\omega; C_\#^\infty(Y))^{n \times n}$ which satisfies

$$\sum_{i,j=1}^n \partial_{y_i y_j} \boldsymbol{\Psi}^{ij} = 0, \ \forall x, y \tag{3.25}$$

and let us calculate

$$\int_\omega \int_Y \boldsymbol{\Phi} \cdot \boldsymbol{\Psi} dy dx = \lim_{h \to 0} \int_\omega \Big(\sum_{i,j=1,\ldots,n} (\partial_j \boldsymbol{v}_h^i(x) - \partial_{ij} u(x)) \cdot \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) \Big) dx$$

$$= -\lim_{h \to 0} \int_\omega \sum_{i,j=1,\ldots,n} (\boldsymbol{v}_h^i(x) - \partial_i u(x)) \partial_{x_j} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx$$

$$- \lim_{h \to 0} \frac{1}{h} \int_\omega \sum_{i,j=1,\ldots,n} (\boldsymbol{v}_h^i(x) - \partial_i u(x)) \partial_{y_j} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx$$

$$\text{using (3.24)} = -\lim_{h \to 0} \frac{1}{h} \int_\omega \sum_{i,j=1,\ldots,n} (\partial_i u^h(x) - \partial_i u(x)) \partial_{y_j} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx$$

$$= -\lim_{h \to 0} \frac{1}{h} \int_\omega \sum_{i,j=1,\ldots,n} (u^h(x) - u(x)) \partial_{y_j x_i} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx$$

$$- \lim_{h \to 0} \frac{1}{h^2} \int_\omega \sum_{i,j=1,\ldots,n} (u^h(x) - u(x)) \partial_{y_i y_j} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx$$

$$\text{using (3.25)} = \lim_{h \to 0} \int_\omega \sum_{i,j=1,\ldots,n} (\partial_j u^h(x) - \partial_j u(x)) \partial_{x_i} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx$$

$$+ \lim_{h \to 0} \int_\omega \sum_{i,j=1,\ldots,n} (u^h(x) - u(x)) \partial_{x_i x_j} \boldsymbol{\Psi}^{ij}(x, \tfrac{x}{h}) dx = 0.$$

Remark 2 In the special case $C = 0$ lemma 3 is just the generalization of Theorem 4. In fact what lemma tells us is that the claim is also valid if we are closer to the gradient than the order of the oscillations.

Lemma 4 *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a convex function which satisfies*

$$|Q(x)| \le C(1 + |x|^2), \quad \forall x \in \mathbb{R}^n, \tag{3.26}$$

for some $C > 0$. Let $(\boldsymbol{u}^h)_{h>0} \subset L^2(\omega; \mathbb{R}^n)$ be a sequence which two-scale converges to $\boldsymbol{u}_0 \in L^2(\omega \times Y; \mathbb{R}^n)$. Then we have that

$$\int_\omega \int_Y Q(\boldsymbol{u}_0(x,y)) dy dx \le \liminf_{h \to 0} \int_\omega Q(\boldsymbol{u}^h(x)) dx \tag{3.27}$$

Proof. Let us take an arbitrary $\boldsymbol{\psi} \in (L^2(\omega; C_{\#}(Y)))^n$. It is well known that if a convex function is finite on an open set than it is continuous. Thus Q is continuous. Also an arbitrary convex function is a pointwise limit of an increasing family of smooth convex Lipschitz functions Q_n. To see this first we use the fact that there exists an increasing family $\widetilde{Q}_n$ of piecewise affine functions (with finitely many cuts) which pointwise converge to Q. Then we define $\widetilde{\widetilde{Q}}_n = \widetilde{Q}_n - \frac{1}{n}$. Finally we smooth every $\widetilde{\widetilde{Q}}_n$ by an appropriate mollifier to preserve the fact that the sequence should be increasing. We obtain for each $n \in \mathbf{N}, h > 0$

$$\int_\omega Q_n(\boldsymbol{u}^h(x))dx \geq \int_\omega Q_n(\boldsymbol{\psi}(x, \frac{x}{h}))dx + \int_\omega DQ_n(\boldsymbol{\psi}(x, \frac{x}{h}))(\boldsymbol{u}^h(x) - \boldsymbol{\psi}(x, \frac{x}{h}))dx. \tag{3.28}$$

By letting $h \to 0$ and using the definition of two-scale convergence and Theorem 2 and the fact that the convexity of Q_n and $Q_n(x) \leq C(1 + |x|^2)$ implies $|DQ_n(x)| \leq C(1 + |x|)$ we obtain for each $n \in \mathbf{N}$

$$\liminf_{h \to 0} \int_\omega Q_n(\boldsymbol{u}^h(x))dx \geq \int_\omega \int_Y Q_n(\boldsymbol{\psi}(x, y))dydx$$
$$+ \int_\omega \int_Y DQ_n(\boldsymbol{\psi}(x, y))(\boldsymbol{u}_0(x, y) - \boldsymbol{\psi}(x, y))dx. \tag{3.29}$$

By using an arbitrariness of $\boldsymbol{\psi}$ and the density of $L^2(\omega; C_{\#}(Y))$ in $L^2(\omega \times Y)$ we conclude that for each $n \in \mathbf{N}$

$$\liminf_{h \to 0} \int_\omega Q_n(\boldsymbol{u}^h(x))dx \geq \int_\omega \int_Y Q_n(\boldsymbol{u}_0(x, y))dydx. \tag{3.30}$$

Since $Q_n < Q$ we conclude

$$\liminf_{h \to 0} \int_\omega Q(\boldsymbol{u}^h(x))dx \geq \int_\omega \int_Y Q_n(\boldsymbol{u}_0(x, y))dydx. \tag{3.31}$$

Letting $n \to \infty$ and using (3.26) we conclude (3.27).

Lemma 5 *Let* $(u_h)_{h>0}$ *be a bounded sequence in* $L^2(\omega)$ *which two-scale converges to* $u_0(x, y) \in L^2(\omega \times Y)$*. Let* $(v_h)_{h>0}$ *be a sequence bounded in* $L^\infty(\omega)$ *which converges in measure to* $v_0 \in L^\infty(\omega)$*. Then* $v_h u_h \rightharpoonup\!\!\!\rightharpoonup v_0(x)u_0(x, y)$*.*
Proof. *We know that* $(v_h u_h)_{h>0}$ *is bounded in* $L^2(\omega)$ *and that there exists a subsequence of* $(v_h)_{h>0}$ *such that* $v_h \to v$ *a.e. in* ω*. Let us take* $\psi \in \mathcal{D}(\omega; C_{\#}(Y))$ *and write,*

$$\int_\omega v_h(x)u_h(x)\psi(x, \frac{x}{h})dx = \int_\omega (v_h(x) - v(x))u_h(x)\psi(x, \frac{x}{h})dx$$
$$+ \int_\omega v(x)u_h(x)\psi(x, \frac{x}{h})dx \tag{3.32}$$

The second converges to

$$\int_\omega \int_Y v(x)u_0(x,y)\psi(x,y)dydx,$$

by the definition of two-scale convergence. We have to prove that the first term in (3.32) converges to 0. By Egoroff's theorem for an arbitrary $\varepsilon > 0$ there exists $E \subset \omega$ such that $\mathrm{meas}(E) < \varepsilon$ and $v_h \to v$ uniformly on E^c. We write

$$\int_\omega (v_h(x) - v(x))u_h(x)\psi(x,\frac{x}{h})dx = \int_E (v_h(x) - v(x))u_h(x)\psi(x,\frac{x}{h})dx$$
$$+ \int_{E^c} (v_h(x) - v(x))u_h(x)\psi(x,\frac{x}{h})dx. \quad (3.33)$$

The second term in (3.33) converges to 0 and can be made arbitrary small. For the first term by the Cauchy inequality we have that there exists $C > 0$ such that

$$\int_E (v_h(x) - v(x))u_h(x)\psi(x,\frac{x}{h})dx \le C\sqrt{\varepsilon}\sup_{h>0}\|u_h\|_{L^2(\omega)}. \quad (3.34)$$

By the arbitrariness of ε we have the claim.

4 Γ-convergence

We shall need the following theorem which can be found in [11].

Theorem 5 (on geometric rigidity) *Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $m \ge 2$. Then there exists a constant $C(U)$ with the following property: for every $\boldsymbol{v} \in W^{1,2}(U;\mathbb{R}^m)$ there is associated rotation $\mathbf{R} \in \mathrm{SO}(m)$ such that*

$$\|\nabla\boldsymbol{v} - \mathbf{R}\|_{L^2(U)} \le C(U)\|\operatorname{dist}(\nabla\boldsymbol{v}, \mathrm{SO}(m)\|_{L^2(U)}. \quad (4.1)$$

The constant $C(U)$ can be chosen uniformly for a family of domains which are Bilipschitz equivalent with controlled Lipschitz constants. The constant $C(U)$ is invariant under dilatations.

In the sequel we suppose $h_0 \ge \frac{1}{2}$ (see Theorem 1). If this was not the case, what follows could be easily adapted. Let us by $P^h : \Omega \to \Omega^h$ denote the map $P^h(x', x_3) = (x', h^2 x_3)$. By ∇_h we denote

$$\nabla_h = \nabla_{\boldsymbol{e}_1, \boldsymbol{e}_2} + \frac{1}{h^2}\nabla_{\boldsymbol{e}_3}.$$

By $\boldsymbol{r}^h : \mathbb{R}^2 \to \mathbb{R}^2$ we denote the mapping $\boldsymbol{r}^h(x_1, x_2) = (\frac{x_1}{h}, \frac{x_2}{h})$. In the same way as in [12, Theorem 10, Remark 11] (see also [18, Lemma 8.1]) we can prove the following theorem. For the adaption we only need Theorem 4.1 and the facts that $C(U)$ can be chosen uniformly for Bilipschitz equivalent domains and that the norms $\|\nabla\boldsymbol{\Theta}^h\|$, $\|(\nabla\boldsymbol{\Theta}^h)^{-1}\|$ are uniformly bounded on Ω^h for $h \le \frac{1}{2}$.

Theorem 6 *Let $\omega \subset \mathbb{R}^2$ be a domain. Let $\boldsymbol{\Theta}^h$ be as above and let $h \leq \frac{1}{2}$. Let $\boldsymbol{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ be such that*

$$\frac{1}{h^2} \int_{\hat{\Omega}^h} \operatorname{dist}^2(\nabla \boldsymbol{y}^h, \mathrm{SO}(3)) dx \leq Ch^8,$$

for some $C > 0$. Then there exists map $\mathbf{R}^h \in W^{1,2}(\omega, \mathrm{SO}(3))$ such that

$$\|(\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{R}^h\|_{L^2(\Omega)} \leq Ch^4, \tag{4.2}$$

$$\|\nabla \mathbf{R}^h\|_{L^2(\omega)} \leq Ch^2. \tag{4.3}$$

Moreover there exist a constant rotation $\bar{\mathbf{Q}}^h \in \mathrm{SO}(3)$ such that

$$\|(\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \bar{\mathbf{Q}}^h\|_{L^2(\Omega)} \leq Ch^2, \tag{4.4}$$

$$\|\mathbf{R}^h - \bar{\mathbf{Q}}^h\|_{L^p(\omega)} \leq C_p h^2, \qquad \forall p < \infty. \tag{4.5}$$

Here all constants depend only on ω (and on p where indicated).

Remark 3 Since $\boldsymbol{\Theta}^h$ is Bilipschitz map, it can easily be seen that the map $\boldsymbol{y} \to \boldsymbol{y} \circ \boldsymbol{\Theta}^h$ is an isomorphism between the spaces $W^{1,2}(\Omega^h; \mathbb{R}^m)$ and $W^{1,2}(\hat{\Omega}^h; \mathbb{R}^m)$ (see e.g. [1]).

To prove Γ-convergence result we need to prove the compactness result, the lower and the upper bound.

4.1 Compactness result

We need the following version of Korn's inequality which is proved in a standard way by contradiction.

Lemma 6 *Let $\omega \subset \mathbb{R}^2$ be a Lipschitz domain. Then there exists $C(\omega) > 0$ such that for an arbitrary $\boldsymbol{u} \in W^{1,2}(\Omega; \mathbb{R}^2)$ we have*

$$\|\boldsymbol{u}\|_{W^{1,2}(\omega;\mathbb{R}^2)} \leq C(\omega)(\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega;\mathbb{R}^2)} + |\int_\omega \boldsymbol{u} dx| + |\int_\omega (\partial_2 \boldsymbol{u}_1 - \partial_1 \boldsymbol{u}_2) dx|). \tag{4.6}$$

Lemma 7 *Let $\boldsymbol{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ be such that*

$$\frac{1}{h^2} \int_{\hat{\Omega}^h} \operatorname{dist}^2(\nabla \boldsymbol{y}^h, \mathrm{SO}(3)) dx \leq Ch^8, \tag{4.7}$$

Then there exists maps $\mathbf{R}^h \in W^{1,2}(\omega, \mathrm{SO}(3))$ and constants $\bar{\mathbf{R}}^h \in \mathrm{SO}(3)$, $\boldsymbol{c}^h \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{y}}^h := (\bar{\mathbf{R}}^h)^T \boldsymbol{y}^h - \boldsymbol{c}^h$$

and the corrected in-plane and the out-of-plane displacements

$$\boldsymbol{u}^h(x_1, x_2) := \frac{1}{h^4}\left(\int_{-1/2}^{1/2}\begin{pmatrix}\widetilde{\boldsymbol{y}}_1^h \circ \boldsymbol{\Theta}^h \circ P^h \\ \widetilde{\boldsymbol{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h\end{pmatrix}(x_1, x_2, x_3)dx_3 - \begin{pmatrix}x_1 \\ x_2\end{pmatrix}\right.$$

$$\left. -h^2\begin{pmatrix}\mathbf{R}_{13}^h(x_1, x_2)\theta_0(\frac{x_1}{h}, \frac{x_2}{h}) \\ \mathbf{R}_{23}^h(x_1, x_2)\theta_0(\frac{x_1}{h}, \frac{x_2}{h})\end{pmatrix}\right),$$

$$v^h(x_1, x_2) := \frac{1}{h^2}\int_{-1/2}^{1/2}(\widetilde{\boldsymbol{y}}_3^h \circ \boldsymbol{\Theta}^h \circ P^h)(x_1, x_2, x_3)dx_3 - \theta(\frac{x_1}{h}, \frac{x_2}{h}) \quad (4.8)$$

satisfy

$$\|(\nabla\widetilde{\boldsymbol{y}}^h)\circ\boldsymbol{\Theta}^h\circ P^h - \mathbf{R}^h\|_{L^2(\Omega)} \le Ch^4, \qquad (4.9)$$

$$\|\mathbf{R}^h - \mathbf{I}\|_{L^p(\omega)} \le C_p h^2 \quad \forall p < \infty, \quad \|\nabla\mathbf{R}^h\|_{L^2(\omega)} \le Ch^2. \qquad (4.10)$$

Moreover every subsequence (not relabeled) has its subsequence (also not relabeled) such that

$$v^h \to v \quad in\ W^{1,2}(\omega), \quad v \in W^{2,2}(\omega) \qquad (4.11)$$

$$\boldsymbol{u}^h \rightharpoonup \boldsymbol{u} \quad in\ W^{1,2}(\omega;\mathbb{R}^2), \qquad (4.12)$$

$$\frac{\mathbf{R}^h - \mathbf{I}}{h^2} \rightharpoonup \mathbf{A} \quad in\ W^{1,2}(\omega;\mathbb{R}^{3\times3}), \qquad (4.13)$$

$$\frac{(\nabla\widetilde{\boldsymbol{y}}^h)\circ\boldsymbol{\Theta}^h\circ P^h - \mathbf{I}}{h^2} \to \mathbf{A} \quad in\ L^2(\Omega;\mathbb{R}^{3\times3}) \qquad (4.14)$$

$$\partial_3\mathbf{A} = 0, \quad \mathbf{A} \in W^{1,2}(\omega;\mathbb{R}^{3\times3}), \qquad (4.15)$$

$$\mathbf{A} = \boldsymbol{e}_3 \otimes \nabla v - \nabla v \otimes \boldsymbol{e}_3, \qquad (4.16)$$

$$\frac{\text{sym}(\mathbf{R}^h - \mathbf{I})}{h^4} \to \frac{\mathbf{A}^2}{2} \quad in\ L^2(\Omega;\mathbb{R}^{3\times3}). \qquad (4.17)$$

Proof. We shall follow the proof of Lemma 13 in [12] (see also Lemma 2 in [24]). Estimates (4.9) and (4.10) follow immediately from Theorem 6 since one can choose $\bar{\mathbf{R}}^h$ so that (4.4) holds with $\bar{\mathbf{Q}}^h = \mathbf{I}$. By applying additional constant in-plane rotation of order h^2 to $\widetilde{\boldsymbol{y}}^h$ and $\mathbf{R}^h$ we may assume in addition to (4.9) and (4.10) that

$$\int_\Omega((\partial_2\widetilde{\boldsymbol{y}}_1^h)\circ\boldsymbol{\Theta}^h\circ P^h - (\partial_1\widetilde{\boldsymbol{y}}_2^h)\circ\boldsymbol{\Theta}^h\circ P^h)dx = 0. \qquad (4.18)$$

By choosing $\boldsymbol{c}^h$ suitably we may also assume that

$$\int_\Omega(\widetilde{\boldsymbol{y}}^h\circ\boldsymbol{\Theta}^h\circ P^h - \boldsymbol{\Theta}^h\circ P^h)dx = 0. \qquad (4.19)$$

Let us define $\mathbf{A}^h = \frac{\mathbf{R}^h - \mathbf{I}}{h^2}$. From (4.10) we get for a subsequence

$$\mathbf{A}^h \rightharpoonup \mathbf{A} \quad in\ W^{1,2}(\omega;\mathbb{R}^{3\times3})$$

Thus we deduce (4.13). Using (2.5), (4.9) we deduce (4.14). Since $(\mathbf{R}^h)^T \mathbf{R}^h = \mathbf{I}$ we have $\mathbf{A}^h + (\mathbf{A}^h)^T = -h^2(\mathbf{A}^h)^T \mathbf{A}^h$. Hence $\mathbf{A} + \mathbf{A}^T = 0$ and after multiplication by $1/h^2$ we obtain (4.17) from the strong convergence of $\mathbf{A}^h$. Using (4.9) we conclude that

$$\| \mathrm{sym}((\nabla \widetilde{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I})\|_{L^2(\Omega)} \le Ch^4. \tag{4.20}$$

The following is useful

$$(\nabla \widetilde{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h = (\nabla(\widetilde{\boldsymbol{y}}^h \circ \boldsymbol{\Theta}^h) \circ P^h)((\nabla \boldsymbol{\Theta}^h)^{-1} \circ P^h)$$
$$= \nabla_h(\widetilde{\boldsymbol{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h)((\nabla \boldsymbol{\Theta}^h)^{-1} \circ P^h). \tag{4.21}$$

From (4.21) it follows

$$((\nabla \widetilde{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\nabla \boldsymbol{\Theta}^h) \circ P^h) = \nabla_h(\widetilde{\boldsymbol{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h). \tag{4.22}$$

Using (2.5), (4.14), (4.22) we conclude that

$$\frac{1}{h^2} \nabla_h(\widetilde{\boldsymbol{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h) \to \mathbf{A} \text{ in } L^2(\Omega; \mathbb{R}^{3 \times 3}). \tag{4.23}$$

From (4.19), (4.23) and the Poincare inequality we conclude the convergence in (4.11). Moreover we have $\partial_i v = \mathbf{A}_{3i}$ for $i = 1, 2$. Hence $v \in W^{2,2}$ since $\mathbf{A} \in W^{1,2}$. Since $\mathbf{A}$ is skew-symmetric we immediately have $\mathbf{A}_{13} = -\partial_1 v$, $\mathbf{A}_{23} = -\partial_2 v$.

By the chain rule the following identities are valid for $i = 1, 2, 3$, $\alpha = 1, 2$,

$$\partial_\alpha(\widetilde{\boldsymbol{y}}_i^h \circ \boldsymbol{\Theta}^h \circ P^h) = (\partial_1 \widetilde{\boldsymbol{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_\alpha \boldsymbol{\Theta}_1^h) \circ P^h$$
$$+ (\partial_2 \widetilde{\boldsymbol{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_\alpha \boldsymbol{\Theta}_2^h) \circ P^h + (\partial_3 \widetilde{\boldsymbol{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_\alpha \boldsymbol{\Theta}_3^h) \circ P^h \tag{4.24}$$

$$\frac{1}{h^2} \partial_3(\widetilde{\boldsymbol{y}}_i^h \circ \boldsymbol{\Theta}^h \circ P^h) = (\partial_1 \widetilde{\boldsymbol{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_3 \boldsymbol{\Theta}_1^h) \circ P^h$$
$$+ (\partial_2 \widetilde{\boldsymbol{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_3 \boldsymbol{\Theta}_2^h) \circ P^h + (\partial_3 \widetilde{\boldsymbol{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_3 \boldsymbol{\Theta}_3^h) \circ P^h \tag{4.25}$$

From (2.3), (4.14), (4.20) we conclude for $\alpha, \beta = 1, 2$

$$\|(\partial_\alpha \widetilde{\boldsymbol{y}}_\alpha^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_\beta \boldsymbol{\Theta}_\beta^h) \circ P^h - (\partial_\beta \boldsymbol{\Theta}_\beta^h) \circ P^h\|_{L^2(\Omega)} \le Ch^4, \tag{4.26}$$

and

$$\|(\partial_3 \widetilde{\boldsymbol{y}}_\alpha^h) \circ \boldsymbol{\Theta}^h \circ P^h \cdot (\partial_\alpha \boldsymbol{\Theta}_3^h) \circ P^h - h\mathbf{R}_{\alpha 3}(\partial_\alpha \theta) \circ \boldsymbol{r}^h\|_{L^2(\Omega)} \le Ch^4, \tag{4.27}$$

for some $C > 0$. Using (4.9), (4.10), (4.14), (4.18), (4.24) we conclude that

$$| \int_\omega (\partial_2 \boldsymbol{u}_1^h - \partial_1 \boldsymbol{u}_2^h) dx | \le Ch^4. \tag{4.28}$$

In the same way using (4.9), (4.10), (4.14), (4.18), (4.24) we conclude that $\|\frac{1}{h^4} \mathrm{sym} \nabla \boldsymbol{u}^h\|_{L^2(\omega)}$ is bounded. Using Lemma 6, (4.19) and (4.28) we have the convergence (4.12). It remains to conclude $\mathbf{A}_{12} = 0$. But this is easy, from (4.12), (4.13) and (4.23).

Lemma 8 *If we additionally assume the following*

$$\left\| \int_{-1/2}^{1/2} \begin{pmatrix} \widetilde{\boldsymbol{y}}_1^h \circ \boldsymbol{\Theta}^h \circ P^h \\ \widetilde{\boldsymbol{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h \end{pmatrix} (x_1, x_2, x_3) dx_3 - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_{L^2(\omega)} \leq Ch^4, \quad (4.29)$$

$$\left\| \int_{-1/2}^{1/2} (\widetilde{\boldsymbol{y}}_3^h \circ \boldsymbol{\Theta}^h \circ P^h)(x_1, x_2, x_3) dx_3 \right\|_{L^2(\omega)} \leq Ch^2, \quad (4.30)$$

we can take $\bar{\mathbf{R}}^h = \mathbf{I}$, $\boldsymbol{c}^h = 0$ *in Lemma 7.*

Proof. First we shall prove that we can take $\bar{\mathbf{Q}}^h = \mathbf{I}$ in Theorem 6. By multiplying (4.4) with $\nabla(\boldsymbol{\Theta}^h \circ P^h)$ and using Poincare inequality on Ω we have that there exists $b^h \in \mathbb{R}^3$ such that

$$\|\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \bar{\mathbf{Q}}^h(\boldsymbol{\Theta}^h \circ P^h) - \mathbf{b}^h\|_{L^2(\Omega)} \leq Ch^2. \quad (4.31)$$

From (4.29), (4.30) and by integrating (4.31) with respect to x_3 we conclude

$$\left\| \bar{\mathbf{Q}}^h \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} - \mathbf{b}^h \right\|_{L^2(\omega)} \leq Ch^2. \quad (4.32)$$

From this, using the fact that $\bar{\mathbf{Q}}^h \in \mathrm{SO}(3)$, we can conclude that $\|\mathbf{b}^h\| \leq Ch^2$ and $\|\bar{\mathbf{Q}}^h - \mathbf{I}\| \leq Ch^2$. Thus we can take $\bar{\mathbf{Q}}^h = \mathbf{I}$ in Theorem 6. Now we conclude that we can repeat the proof of Lemma 7 with assumptions (4.29) and (4.30) instead of assumptions (4.18) (i.e. (4.28)) and (4.19).

Lemma 9 *Let* $\widetilde{\boldsymbol{y}}^h, \mathbf{R}^h, \boldsymbol{u}^h, \boldsymbol{u}, v^h, v$ *be as in Lemma 7. Then there exist unique* $\boldsymbol{u}_1 \in L^2(\omega; \dot{H}_\#^1(Y))^2$ *and* $v_1 \in L^2(\omega; \dot{H}_\#^2(Y))$ *such that*

$$\nabla \boldsymbol{u}^h \rightharpoonup\!\!\!\rightharpoonup \nabla_x \boldsymbol{u}(x) + \nabla_y \boldsymbol{u}_1(x, y) \ in \ L^2(\omega \times Y; \mathbb{R}^{2 \times 2}), \quad (4.33)$$

$$\nabla \frac{1}{h^2} \begin{pmatrix} \mathbf{R}_{13}^h \\ \mathbf{R}_{23}^h \end{pmatrix} \rightharpoonup\!\!\!\rightharpoonup -\nabla_x^2 v(x) - \nabla_y^2 v_1(x, y) \ in \ L^2(\omega \times Y; \mathbb{R}^{2 \times 2}). \quad (4.34)$$

Proof. The relation (4.33) is the direct consequence of the relation (4.12) and Theorem 4. To prove (4.34) we shall use Lemma 2. From the relation (4.9) using the boundedness of $\nabla\boldsymbol{\Theta}^h$ and (4.22) we conclude

$$\|\frac{1}{h^2} \int_{-1/2}^{1/2} (\nabla_h(\widetilde{\boldsymbol{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h) dx_3$$

$$-\frac{1}{h^2}(\mathbf{R}^h - \mathbf{I}) \int_{-1/2}^{1/2} \nabla_h(\boldsymbol{\Theta}^h \circ P^h) dx_3\|_{L^2(\omega)} \leq Ch^2. \quad (4.35)$$

From (4.35) using (2.3) and (4.17) we conclude

$$\|\partial_1 v^h - \frac{1}{h^2}\mathbf{R}_{31}^h\|_{L^2(\omega)} \leq Ch^2, \ \|\partial_2 v^h - \frac{1}{h^2}\mathbf{R}_{32}^h\|_{L^2(\omega)} \leq Ch^2, \quad (4.36)$$

for some $C > 0$. From that we have, by using (4.17),

$$\|\partial_1 v^h + \frac{1}{h^2}\mathbf{R}_{13}^h\|_{L^2(\omega)} \leq Ch^2, \ \|\partial_2 v^h + \frac{1}{h^2}\mathbf{R}_{23}^h\|_{L^2(\omega)} \leq Ch^2. \quad (4.37)$$

The claim is now the direct consequence of Lemma 2 and (4.13).

4.2 Lower bound

Remark 4 In the next lemma we shall characterize the limiting strain and express it in terms of $\boldsymbol{u}, \boldsymbol{u}_1, v, v_1, \theta$ (see Lemma 9). Since the strain is an element of $L^2(\Omega)$ and we want to obtain its two-scale limit which characterizes only oscillations of the first two variables of the order h the following modification of Definition 1) is needed: A sequence $(u_h)_{h>0}$ of functions in $L^2(\Omega)$ converges two-scale to a function u_0 belonging to $L^2(\Omega \times Y)$ if for every $\psi \in L^2(\Omega; C_\#(Y))$,

$$\int_\Omega u_h(x_1, x_2, x_3)\psi(x_1, x_2, x_3, \frac{x_1}{h}, \frac{x_2}{h}) \to \int_\Omega \int_Y u_0(x_1, x_2, x_3, y)\psi(x_1, x_2, x_3, y).$$

Here $Y = [0,1]^2$. It can be also seen that the analogous statements of Theorem 2, Theorem 3, Theorem 4 and Remark 1 are valid (see [22] for time dependent problems). Also the analogous conclusions of Lemma 1 Lemma 2, Lemma 3, Lemma 4 and Lemma 5 are valid (∇_x should be replaced by the gradient in the first two variables).

Lemma 10 *Consider $\boldsymbol{y}^h : \hat{\Omega}^h \to \mathbb{R}^3$, $\mathbf{R}^h \in W^{1,2}(\omega; \mathrm{SO}(3))$ and define $\boldsymbol{u}^h, v$ by (4.8) Suppose that we have a subsequence of $\boldsymbol{y}^h$ such that (4.9)-(4.17) are valid. Additionally we suppose*

$$\nabla \boldsymbol{u}^h \rightharpoonup\rightharpoonup \nabla_x \boldsymbol{u}(x) + \nabla_y \boldsymbol{u}_1(x,y) \ in \ L^2(\omega \times Y; \mathbb{R}^{2\times 2}), \tag{4.38}$$

$$\nabla \frac{1}{h^2}\begin{pmatrix} \mathbf{R}_{13}^h \\ \mathbf{R}_{23}^h \end{pmatrix} \rightharpoonup\rightharpoonup -\nabla_x^2 v(x) - \nabla_y^2 v_1(x,y) \ in \ L^2(\omega \times Y; \mathbb{R}^{2\times 2}), \tag{4.39}$$

for $\boldsymbol{u}_1 \in L^2(\omega; \dot{H}_\#^1(Y))^2$ and $v_1 \in L^2(\omega; \dot{H}_\#^2(Y))$. Then

$$\mathbf{G}^h := \frac{(\mathbf{R}^h)^T((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h) - \mathbf{I}}{h^4} \rightharpoonup\rightharpoonup \mathbf{G} \quad in \ L^2(\Omega \times Y; \mathbb{R}^{3\times 3}), \tag{4.40}$$

and the 2×2 sub-matrix $\mathbf{G}''$ given by $\mathbf{G}''_{\alpha\beta} = \mathbf{G}_{\alpha\beta}$ for $1 \le \alpha, \beta \le 2$ satisfies

$$\mathbf{G}''(x', x_3) = \mathbf{G}_0(x_1, x_2, y) + x_3 \mathbf{G}_1(x_1, x_2, y), \tag{4.41}$$

where

$$\mathrm{sym}\, \mathbf{G}_0(x_1, x_2, y) = \mathrm{sym}\, \nabla_x \boldsymbol{u}(x) + \mathrm{sym}\, \nabla_y \boldsymbol{u}_1(x,y)$$
$$-\nabla_x^2 v(x)\theta_0(y) - \nabla_y^2 v_1(x,y)\theta_0(y) + \frac{1}{2}\nabla_x v \otimes \nabla_x v \tag{4.42}$$
$$\mathbf{G}_1(x_1, x_2, y) = -(\nabla_x)^2 v(x) - (\nabla_y)^2 v_1(x,y). \tag{4.43}$$

Proof. We follow the proof of Lemma 15 in [12] (see also Lemma 4 in [24]). By the assumption $\mathbf{G}^h$ is bounded in L^2, thus a subsequence converges weakly.

To show that the limit matrix $\mathbf{G}''$ is affine in x_3 we consider the difference quotients

$$\mathbf{H}^h(x', x_3) = s^{-1}[\mathbf{G}^h(x_1, x_2, x_3 + s) - \mathbf{G}^h(x_1, x_2, x_3)]. \tag{4.44}$$

By multiplying the definition of $\mathbf{G}^h$ with $\mathbf{R}^h$ and using (4.22) we obtain for $\alpha, \beta \in \{1, 2\}$

$$
\begin{aligned}
(\mathbf{R}^h \mathbf{H}^h)_{\alpha\beta} &= \frac{1}{sh^4} [\nabla_h (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h)(x', x_3 + s) - \\
&\qquad - \nabla_h (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h)(x', x_3)]_{\alpha\beta} \\
&\quad + \frac{1}{sh^4} \Big[((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)(x', x_3 + s) \cdot \\
&\qquad \cdot \Big(\mathbf{I} - ((\nabla \boldsymbol{\Theta}^h) \circ P^h)(x', x_3 + s) \Big) \Big]_{\alpha\beta} \\
&\quad - \frac{1}{sh^4} \Big[((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)(x', x_3) \cdot \\
&\qquad \cdot \Big(\mathbf{I} - ((\nabla \boldsymbol{\Theta}^h) \circ P^h)(x', x_3) \Big) \Big]_{\alpha\beta} \\
&= \frac{1}{sh^4} \Big[(\nabla_h (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)(x', x_3 + s) - \\
&\qquad - \nabla_h (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)(x', x_3) \Big]_{\alpha\beta} \\
&\quad + \frac{1}{sh^4} \Big[((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)(x', x_3 + s) - \\
&\qquad - \mathbf{R}^h(x')) \Big(\mathbf{I} - ((\nabla \boldsymbol{\Theta}^h) \circ P^h)(x', x_3 + s) \Big) \Big]_{\alpha\beta} \\
&\quad - \frac{1}{sh^4} \Big[((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)(x', x_3) \\
&\qquad - \mathbf{R}^h(x')) \Big(\mathbf{I} - ((\nabla \boldsymbol{\Theta}^h) \circ P^h)(x', x_3) \Big) \Big]_{\alpha\beta} \\
&\quad - \frac{1}{sh^4} \Big[(\mathbf{R}^h(x') - \mathbf{I})((\nabla \boldsymbol{\Theta}^h) \circ P^h)(x', x_3 + s) - \\
&\qquad - ((\nabla \boldsymbol{\Theta}^h) \circ P^h)(x', x_3) \Big]_{\alpha\beta}.
\end{aligned}
\tag{4.45}
$$

By using (2.3), (4.9) we conclude that the second and third term converges to 0 strongly in $L^2(\omega \times (-\frac{1}{2}, \frac{1}{2} - s)$. The forth term also converges to 0 by (2.3) and (4.13). Thus we have that the first term in (4.45) is bounded in $L^2(\Omega)$ and thus two scale converges to some $\mathbf{G}_1 \in L^2(\Omega \times Y)$. We conclude

$$
\begin{aligned}
\frac{1}{sh^4} [(\nabla_h (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)(x', x_3 + s) \\
- \nabla_h (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)(x', x_3))]_{\alpha\beta} \\
= \frac{1}{h^2} \partial_\beta \Big(\frac{1}{s} \int_0^s \frac{1}{h^2} \partial_3 (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)_\alpha \Big).
\end{aligned}
\tag{4.46}
$$

By using (4.20) (which is a consequence of (4.9) and (4.17)) and (4.25) we conclude that there exists $C > 0$ such that for $\alpha = 1, 2$, $\beta = 3 - \alpha$

$$
\begin{aligned}
\Big\| \Big(\frac{1}{sh^2} \int_0^s \frac{1}{h^2} \partial_3 (\boldsymbol{y}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)_\alpha \Big) - \frac{1}{h^2} \mathbf{R}_{\alpha 3}^h \\
+ \frac{1}{h} \mathbf{R}_{\alpha\beta}^h (\partial_\beta \theta) \circ \boldsymbol{r}^h \Big\|_{L^2(\omega \times (-\frac{1}{2}, \frac{1}{2} - s))} < C h^2.
\end{aligned}
\tag{4.47}
$$

By using Lemma 1, the fact that for $\alpha = 1, 2$, $\beta = 3 - \alpha$

$$\frac{1}{h}\mathbf{R}^h_{\alpha\beta}(\partial_\beta\theta) \circ \boldsymbol{r}^h \to 0 \text{ in } W^{1,2}(\omega) \tag{4.48}$$

(this follows from (4.13)), we conclude from (4.45) that

$$\mathbf{R}^h\mathbf{H}^h \rightharpoonup\!\!\!\rightharpoonup \mathbf{G}_1, \quad \text{in } L^2(\omega \times (-\frac{1}{2}, \frac{1}{2} - s); \mathbb{R}^{n \times n}), \tag{4.49}$$

where $\mathbf{G}_1$ is given by (4.43). Since $\mathbf{R}^h \to \mathbf{I}$ boundedly a.e. we conclude $\mathbf{H}^h \rightharpoonup\!\!\!\rightharpoonup \mathbf{G}_1$ from Lemma 5. From this we have also (4.41). In order to prove formula for $\mathbf{G}_0$ it suffices to study

$$\mathbf{G}^h_0(x') = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{G}^h(x', x_3)dx_3.$$

We have for $\alpha, \beta \in \{1, 2\}$

$$(\mathbf{G}^h)_{\alpha\beta}(x', x_3) = \frac{((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I})_{\alpha\beta}}{h^4} - \frac{(\mathbf{R}^h - \mathbf{I})_{\alpha\beta}}{h^4}$$
$$+ \left[(\mathbf{R}^h - \mathbf{I})^T \frac{(\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{R}^h}{h^4} \right]_{\alpha\beta}. \tag{4.50}$$

The third term in (4.50) converges strongly to 0 in $L^2(\Omega)$. From (4.24) we conclude, after a little calculation by using (2.3), (4.9) and (4.13), that

$$\nabla \boldsymbol{u}^h - \int_{-1/2}^{1/2} \frac{((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I})}{h^4} dx_3$$
$$+ (\theta_0 \circ \boldsymbol{r}^h)\nabla \frac{1}{h^2} \begin{pmatrix} \mathbf{R}^h_{13} \\ \mathbf{R}^h_{23} \end{pmatrix} \to 0 \text{ in } L^2(\Omega; \mathbb{R}^{2 \times 2}). \tag{4.51}$$

From (4.51) we conclude

$$\int_{-1/2}^{1/2} \frac{((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I})}{h^4} dx_3 \rightharpoonup\!\!\!\rightharpoonup \nabla_x \boldsymbol{u}(x) + \nabla_y \boldsymbol{u}_1(x, y)$$
$$- (\nabla_x)^2 v(x)\theta_0(y) - (\nabla_y)^2 v_1(x, y)\theta_0(y) \text{ in } L^2(\omega \times Y; \mathbb{R}^{2 \times 2}). \tag{4.52}$$

From (4.51) and (4.52) we have (4.42).

By $I^L : W^{1,2}(\omega; \mathbb{R}^2) \times L^2(\omega; \dot{H}^1_\#(Y))^2 \times W^{2,2}(\omega) \times L^2(\omega; \dot{H}^2_\#) \to \mathbb{R}^+_0$ we denote the functional

$$I^L(\boldsymbol{u}, \boldsymbol{u}_1, v, v_1) = \int_\omega \int_Y \Big(\frac{1}{2}Q_2\big(\text{sym}\,\nabla_x\boldsymbol{u}(x) + \text{sym}\,\nabla_y\boldsymbol{u}_1(x, y) - \nabla^2_x v(x)\theta_0(y)$$
$$- \nabla^2_y v_1(x, y)\theta_0(y) + \frac{1}{2}\nabla_x v \otimes \nabla_x v\big)\Big)dydx$$
$$+ \frac{1}{24}\int_\omega \int_Y Q_2(\nabla^2_y v_1(x, y))dydx + \frac{1}{24}\int_\omega Q_2(\nabla^2_x v(x))dx. \tag{4.53}$$

Corollary 1 *Let* $\boldsymbol{y}^h, \mathbf{R}^h, \boldsymbol{u}^h, \boldsymbol{u}, \boldsymbol{u}_1, v^h, v, v_1, \mathbf{G}^h, \mathbf{G}, \mathbf{G}'', \mathbf{G}_0, \mathbf{G}_1$ *be as in Lemma 10. Then we have the following semi-continuity results.*

$$\liminf_{h \to 0} \frac{1}{h^8} I^h(\boldsymbol{y}^h) \geq I^L(\boldsymbol{u}, \boldsymbol{u}_1, v, v_1). \tag{4.54}$$

Proof. We shall use the truncation, the Taylor expansion and the weak semi-continuity argument as in the proof of Corollary 16 in [12]. Let $m : [0, \infty) \to [0, \infty)$ denote a modulus of continuity of $D^2 W$ near the identity and consider the good set $\Omega_h := \{x \in \Omega : |\mathbf{G}^h(x)| < h^{-1}\}$. Its characteristic function χ_h is bounded and satisfies $\chi_h \to 1$ a.e. in Ω. Thus we have $\chi_h \mathbf{G}^h \rightharpoonup\rightharpoonup \mathbf{G}$ in $L^2(\Omega \times Y; \mathbb{R}^{3\times3})$ by Lemma 5. By Taylor expansion

$$\frac{1}{h^8} \chi_h W(\mathbf{I} + h^4 \mathbf{G}^h) \geq \frac{1}{2} Q_3(\chi_h \mathbf{G}^h) - m(h^3)|\mathbf{G}^h|^2. \tag{4.55}$$

Using i), ii), (4.55), the boundedness of sequence $\mathbf{G}^h$ in $L^2(\Omega; \mathbb{R}^{3\times3})$ and Lemma 4 we conclude

$$\liminf_{h \to 0} \frac{1}{h^8} I^h(\boldsymbol{y}^h) = \liminf_{h \to 0} \frac{1}{h^8} \int_\Omega W((\mathbf{R}^h)^T((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)) dx$$

$$\geq \liminf_{h \to 0} \left[\frac{1}{2} \int_\Omega Q_3(\chi_h \mathbf{G}^h) dx + \frac{1}{h^8} \int_\Omega (1 - \chi_h) W((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)) dx \right]$$

$$\geq \frac{1}{2} \int_\Omega \int_Y Q_3(\mathbf{G}(x, y)) dy dx \geq \frac{1}{2} \int_\Omega \int_Y Q_2(\mathbf{G}''(x, y)) dy dx. \tag{4.56}$$

Now by (4.41) we have

$$\int_{-1/2}^{1/2} \int_Y Q_2(\mathbf{G}''(x_1, x_2, x_3, y)) dy dx_3 = \int_Y Q_2(\mathbf{G}_0(x_1, x_2, y)) dy$$

$$+ \frac{1}{12} \int_Y Q_2(\mathbf{G}_1(x_1, x_2, y)) dy. \tag{4.57}$$

This implies the claim of the corollary.

Let us by $Q_2^H : \mathbb{R}_{\text{sym}}^{2\times2} \times \mathbb{R}_{\text{sym}}^{2\times2} \to \mathbb{R}_0^+$ denote the functional

$$Q_2^H(\mathbf{G}, \mathbf{F}) = \min_{\substack{\boldsymbol{u}_1 \in \dot{H}_\#^1(Y), \\ v \in \dot{H}_\#^2(Y)}} I_{\mathbf{G}, \mathbf{F}}^H, \tag{4.58}$$

where we have by $I_{\mathbf{G}, \mathbf{F}}^H : (\dot{H}_\#^1(Y))^2 \times \dot{H}_\#^2(Y) \to \mathbb{R}_0^+$ denoted the functional

$$I_{\mathbf{G}, \mathbf{F}}^H(\boldsymbol{u}_1, v_1) = \int_Y \left(Q_2(\mathbf{G} + \mathbf{F}\theta_0 + \text{sym} \nabla \boldsymbol{u}_1(y) - \nabla^2 v_1(y)\theta_0(y)) \right.$$

$$\left. + \frac{1}{12} Q_2(\nabla^2 v_1(y)) \right) dy \tag{4.59}$$

By $I_0^L : W^{1,2}(\omega; \mathbb{R}^2) \times W^{2,2}(\omega) \to \mathbb{R}_0^+$ we denote the functional

$$I_0^L(\boldsymbol{u}, v) = \frac{1}{2} \int_\omega Q_2^H(\text{sym} \nabla \boldsymbol{u} + \frac{1}{2} \nabla v \otimes \nabla v, -\nabla^2 v) dx + \frac{1}{24} \int_\omega Q_2(\nabla^2 v) dx. \tag{4.60}$$

In the sequel we shall analyze the property of Q_2^H.

Lemma 11 *For every* $\mathbf{G}, \mathbf{F} \in \mathbb{R}^{2\times 2}$ *there exists a unique* $\boldsymbol{u}_1(\mathbf{G}, \mathbf{F}) \in (\dot{H}^1_{\#}(Y))^2$, $v_1(\mathbf{G}, \mathbf{F}) \in \dot{H}^2_{\#}(Y)$ *which minimizes the functional* $I^H_{\mathbf{G}, \mathbf{F}}$. *This minimizer satisfies the estimate*

$$\|\boldsymbol{u}_1(\mathbf{G}, \mathbf{F})\|^2_{\dot{H}^1_{\#}(Y)}, \|v_1(\mathbf{G}, \mathbf{F})\|^2_{\dot{H}^2_{\#}(Y)} \leq C(\|\mathbf{G}\|^2 + \|\mathbf{F}\|^2), \qquad (4.61)$$

for some $C > 0$. *The functional* Q^H_2 *is a nonnegative quadratic form. Let us by* V *denote the subspace of* $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$

$$V = \left\{\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}; \forall y \in Y, a_{11}\partial_{22}\theta(y) + a_{22}\partial_{11}\theta(y) - 2a_{12}\partial_{12}\theta(y) = 0\right\},$$

and by $V^\perp$ *its orthogonal complement in* $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$. *Then there exists* $C > 0$ *such that*

$$Q^H_2(\mathbf{G}, \mathbf{F}) > C(\|\mathbf{G}\|^2 + \|\mathbf{F}^\perp\|^2), \forall \mathbf{G}, \mathbf{F} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \qquad (4.62)$$

where $\mathbf{F}^\perp \in V^\perp$ *is the unique matrix such that* $\mathbf{F} - \mathbf{F}^\perp \in V$.

Proof. The existence of the minimum of the functional $I^H_{\mathbf{G}, \mathbf{F}}$ in the space $(\dot{H}^1_{\#}(Y))^2 \times \dot{H}^2_{\#}(Y)$ is guaranteed by the following facts:

1. for each $\mathbf{G}, \mathbf{F}$ the functional $I^H_{\mathbf{G}, \mathbf{F}}$ is sequentially weakly lower semi continuous by the convexity of the form Q_2.
2. for each $\mathbf{G}, \mathbf{F}$ the functional $I^H_{\mathbf{G}, \mathbf{F}}$ is coercive in the sense

$$\|\boldsymbol{u}^n_1\|_{(\dot{H}^1_{\#}(Y))^2} \to +\infty \text{ or } \|v^n_1\|_{\dot{H}^2_{\#}(Y)} \to +\infty \Rightarrow I^H_{\mathbf{G}, \mathbf{F}}(\boldsymbol{u}^n_1, v^n_1) \to +\infty.$$

To prove this let us note that the boundedness of $I^H_{\mathbf{G}, \mathbf{F}}(\boldsymbol{u}^n_1, v^n_1)$ directly implies the boundedness of $\|v^n_1\|_{\dot{H}^2_{\#}(Y)}$. This, on the other hand, implies the boundedness of $\|\operatorname{sym}\nabla\boldsymbol{u}^n_1\|_{L^2(Y)}$. From Korn's inequality we have the claim.

The uniqueness of the minimum is the consequence of the strict convexity of the form Q_2 on symmetric matrices.

Let us by $\mathcal{A}: \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ denote the positive definite linear operator which realizes the quadratic functional Q_2 i.e. we have

$$Q_2(\mathbf{G}) = (\mathcal{A}\mathbf{G}, \mathbf{G}), \forall \mathbf{G} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \qquad (4.63)$$

where we have by $(\cdot, \cdot)$ denoted the standard scalar multiplication on $\mathbb{R}^{2\times 2}$. The minimization formulation (4.58) implies

$$\int_Y (\mathcal{A}(\operatorname{sym}\nabla\boldsymbol{u}_1(y) - \nabla^2 v_1(y)\theta_0(y)), \operatorname{sym}\nabla\boldsymbol{\varphi}(y))dy$$

$$+ \int_Y (\mathcal{A}(\operatorname{sym}\nabla\boldsymbol{u}_1(y) - \nabla^2 v_1(y)\theta_0(y)), -\nabla^2 \upsilon(y)\theta_0(y))dy$$

$$+ \frac{1}{6}\int_Y (\mathcal{A}\nabla^2 v_1(y), \nabla^2 \upsilon(y))dy$$

$$= -\int_Y (\mathcal{A}\mathbf{G}, \nabla^2 \upsilon(y)\theta_0(y))dy + \int_Y (\mathcal{A}\mathbf{F}\theta_0(y), \operatorname{sym}\nabla\boldsymbol{\varphi}(y) - \nabla^2 \upsilon(y)\theta_0(y))dy$$

$$\forall \boldsymbol{\varphi} \in (\dot{H}^1_{\#}(Y))^2, \upsilon \in \dot{H}^2_{\#}(Y) \quad (4.64)$$

From (4.64) it can be easily seen that $\boldsymbol{u}_1(\mathbf{G}, \mathbf{F})$, $v_1(\mathbf{G}, \mathbf{F})$ depends linearly on $(\mathbf{G}, \mathbf{F})$ and thus Q_2^H is a nonnegative quadratic form. To see the estimate (4.61) we just take $\boldsymbol{\varphi} = \boldsymbol{u}_1$, $v = v_1$. We obtain

$$\|\boldsymbol{u}_1 - \nabla^2 v_1 \theta_0\|_{L^2(Y)} + \|\nabla^2 v_1\|_{L^2(Y)} \leq C(\|\mathbf{G}\| + \|\mathbf{F}\|). \qquad (4.65)$$

From (4.65) we have (4.61). To check the positive definiteness of Q_2^H we have to check

$$Q_2^H(\mathbf{G}, \mathbf{F}) = 0, \mathbf{G}, \mathbf{F} \in \mathbb{R}_{\text{sym}}^{2\times 2} \Rightarrow \mathbf{G} = 0, \ \mathbf{F} \in V. \qquad (4.66)$$

Let us suppose $Q_2^H(\mathbf{G}, \mathbf{F}) = 0$. From (4.59) we conclude that

$$v_1(y) = 0, \quad \mathbf{G} + \mathbf{F}\theta_0(y) + \operatorname{sym} \nabla \boldsymbol{u}_1(y) = 0, \ \forall y \in Y. \qquad (4.67)$$

Integrating (4.67) in Y and using the fact $\mathbf{u}_1 \in \dot{H}_{\#}^1(Y)$ we conclude $\mathbf{G} = 0$.

From the second equation we conclude that $\mathbf{F}\theta_0$ is symmetrized gradient and this implies

$$\mathbf{F}_{11}\partial_{22}\theta + \mathbf{F}_{22}\partial_{11}\theta - 2\mathbf{F}_{12}\partial_{12}\theta = 0. \qquad (4.68)$$

From this we have that $\mathbf{F} \in V$.

Now we shall say something about the regularity of (4.64). First we will need one technical lemma which is a variant of well known lemma (see e.g. [10]) for torus.

Lemma 12 *Let us take $u \in L^2(Y)$ and let us extend u to $\mathbb{R}^2$ by periodicity. Define the i-th difference quotient of size h by*

$$D_i^h(u)(y) = \frac{u(y + h\boldsymbol{e}_i) - u(y)}{h}, \ (i = 1, 2) \qquad (4.69)$$

for $y \in Y$ and $h \in \mathbb{R}$ and let us define

$$D^h(u) := (D_1^h u, D_2^h u). \qquad (4.70)$$

We have:

i) Suppose $u \in H_{\#}^1(Y)$. Then

$$\|D^h u\|_{L^2(Y)} \leq C\|Du\|_{L^2(Y)},$$

   for some $C > 0$ and all $0 < |h| < h_0$.
ii) Assume $u \in L^2(Y)$ and there exists a constant C such that

$$\|D^h u\|_{L^2(Y)} \leq C,$$

   for all $0 < |h| < h_0$. Then

$$u \in H_{\#}^1(Y), \ with \ \|Du\|_{L^2(Y)} \leq C.$$

Proof. Let us take $Y' = [-1, 2]^2$.

i) From standard theorem on difference quotients we conclude that there exists $C_1 >$ such that

$$\|D^h u\|_{L^2(Y)} \le C_1 \|Du\|_{L^2(Y')}, \qquad (4.71)$$

for all $0 < |h| < 1$. Since, by periodicity, $\|Du\|_{L^2(Y')} = 9\|Du\|_{L^2(Y)}$ we have the claim.

ii) Let us take $u \in L^2(Y)$ and extend it, by periodicity to $L^2(Y')$. To conclude that $\|Du\|_{L^2(Y)} \le C$ is the direct consequence of the standard theorem on difference quotient. We have to prove $u \in H^1_\#(Y)$, i.e. it has periodic boundary conditions. But this can be concluded from the fact that $\|D^h u\|_{L^2(V)} \le 9C$, for every V open $Y \subset V \subset Y'$. This implies $\|Du\|_{L^2(V)} \le 9C$, by standard theorem on difference quotients, and thus $u \in H^1(V)$. This, on the other hand, implies that u has periodic boundary conditions.

Lemma 13 *Let $\theta \in C^k_\#(Y)$. The solution $\boldsymbol{u}_1, v_1$ of (4.64) is a linear function of $\mathbf{G}, \mathbf{F}$ and $\boldsymbol{u}_1$ is in the space $\dot{H}^{k+1}_\#(Y)$ and v_1 is in the space $\dot{H}^{k+2}_\#(Y)$.*

Proof.   Since we are on torus, proving regularity is easier since we do not have boundary. Let us prove the claim for $k = 1$. Let us take $i \in \{1, 2\}$ and test functions

$$\boldsymbol{\varphi}_\alpha = -D_i^{-h}(D_i^h \boldsymbol{u}_\alpha), \quad \alpha = 1, 2;\ v = -D_i^{-h}(D_i^h v_1)$$

in (4.64). From standard properties on difference quotient we conclude

$$\int_Y (\mathcal{A}(\operatorname{sym}\nabla D_i^h \boldsymbol{u}_1(y) - \nabla^2 D_i^h v_1(y)\theta_0(y)), \operatorname{sym}\nabla D_i^h \boldsymbol{u}_1)dy$$

$$+ \int_Y (\mathcal{A}(\operatorname{sym}\nabla D_i^h \boldsymbol{u}_1(y) - \nabla^2 D_i^h v_1(y)\theta_0(y)), -\nabla^2 D_i^h v_1(y)\theta_0(y))dy$$

$$+ \frac{1}{6}\int_Y (\mathcal{A}\nabla^2 D_i^h v_1(y), \nabla^2 D_i^h v_1(y))dy$$

$$= -\int_Y (\mathcal{A}\mathbf{G}, \nabla^2 D_i^h v_1(y)\theta_0(y))dy$$

$$+ \int_Y (\mathcal{A}\mathbf{F}\theta_0(y), \operatorname{sym}\nabla D_i^h \boldsymbol{u}_1(y) - \nabla^2 D_i^h v_1(y)\theta_0(y))dy$$

$$- \int_Y (\mathcal{A}(\operatorname{sym}\nabla \boldsymbol{u}_1(y) - \nabla^2 v_1(y)D_i^h \theta_0(y)), \operatorname{sym}\nabla \boldsymbol{u}_1(y))dy$$

$$- \int_Y (\mathcal{A}(\operatorname{sym}\nabla \boldsymbol{u}_1(y) - \nabla^2 v_1(y)\theta_0(y)), -\nabla^2 v_1(y)D_i^h \theta_0(y))dy$$

$$- \int_Y (\mathcal{A}(\operatorname{sym}\nabla \boldsymbol{u}_1(y) - \nabla^2 v_1(y)D_i^h \theta_0(y)), -\nabla^2 v_1(y)\theta_0(y))dy \qquad (4.72)$$

From (4.72), using the positive definiteness of $\mathcal{A}$ and transposing the different quotient in the first two terms on the right hand side, we conclude that there

exists $C > 0$, dependent only on $\mathcal{A}$ such that

$$\| \operatorname{sym} \nabla D_i^h \boldsymbol{u}_1 - \nabla^2 D_i^h v_1 \theta_0) \|_{L^2(Y)}^2 + \| \nabla^2 D_i^h v_1 \|_{L^2(Y)}^2$$
$$\leq C(\|\mathbf{G}\|^2 + \|\mathbf{F}\|^2 + \|\theta_0\|_{C^1(Y)}^2)(\|\boldsymbol{u}_1\|_{\dot{H}_\#^1(Y)}^2 + \|v_1\|_{\dot{H}_\#^2(Y)}^2). \quad (4.73)$$

From this we conclude that $\|D_i^h \boldsymbol{u}_1\|_{H^1(Y)}$, $\|D_i^h v_1\|_{H^2(Y)}$ are bounded, independent of h, for $i \in \{1, 2\}$. Using lema 12 we have the claim for $k = 1$. For general k we just differentiate (4.64) and repeat the arguments (see e.g. [10]). Thus we have that the solution of (4.64) for $\theta \in C_\#^k(Y)$ is given by

$$\boldsymbol{u}_1^\alpha(y) = \mathbf{A}_u^\alpha(y) \cdot \mathbf{G} + \mathbf{B}_u^\alpha(y) \cdot \mathbf{F}, \ \alpha = 1, 2, \ v_1(y) = \mathbf{A}_v(y) \cdot \mathbf{G} + \mathbf{B}_v(y) \cdot \mathbf{F}, \quad (4.74)$$

where $\mathbf{A}_u^\alpha, \mathbf{B}_u^\alpha \in H_\#^{k+1}(Y; \mathbb{R}^{2\times2})$, $\mathbf{A}_v, \mathbf{B}_v \in H_\#^{k+2}(Y; \mathbb{R}^{2\times2})$.

4.3 Upper bound

Theorem 7 (optimality of lower bound) *Let* $v \in W^{2,2}(\omega)$, $\boldsymbol{u} \in W^{1,2}(\omega; \mathbb{R}^2)$ *and let* $\theta \in C_\#^2(Y)$. *Then for each sequence* $h \to 0$ *there exists a subsequence, still denoted by* h *and appropriate* $\boldsymbol{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ *and* $\mathbf{R}^h \in W^{1,2}(\omega; \operatorname{SO}(3))$ *such that*

$$\|(\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{R}^h\|_{L^2(\Omega)} \leq Ch^4, \quad (4.75)$$

$$\|\nabla \mathbf{R}^h\|_{L^2(\omega)} \leq Ch^2, \frac{\mathbf{R}^h - \mathbf{I}}{h^2} \rightharpoonup \mathbf{A} \quad in \ W^{1,2}(\omega; \mathbb{R}^{3\times3}), \quad (4.76)$$

where $\mathbf{A}$ *is defined by (4.16) and for* $\boldsymbol{u}^h$, v^h *defined by (4.8) (where* $\widetilde{\boldsymbol{y}}$ *should be replaced by* $\boldsymbol{y}$*) convergence (4.11)-(4.12) are valid and*

$$\lim_{h\to0} I^h(\boldsymbol{y}^h) = I_0^L(\boldsymbol{u}, v). \quad (4.77)$$

Proof. Let us assume that $\boldsymbol{u} \in C^\infty(\omega; \mathbb{R}^2), v \in C^\infty(\omega)$. Let us take an arbitrary $\boldsymbol{u}_1 \in \mathcal{D}(\omega; C_\#^\infty(Y))^2$, $v_1 \in \mathcal{D}(\omega; C_\#^\infty(Y))$. Then we define

$$\boldsymbol{y}^h(\boldsymbol{\Theta}^h(x_1, x_2, x_3^h)) = \boldsymbol{\Theta}^h(x_1, x_2, x_3)$$
$$+ \begin{pmatrix} h^4 \boldsymbol{u}(x_1, x_2) + h^5 \boldsymbol{u}_1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}, x_3) \\ h^2 v(x_1, x_2) + h^4 v_1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}, x_3) \end{pmatrix}$$
$$-h^2 x_3^h \begin{pmatrix} \partial_1 v(x') + h \partial_{y_1} v_1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}, x_3) \\ \partial_2 v(x') + h \partial_{y_2} v_1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}, x_3) \\ 0 \end{pmatrix}$$
$$-h^4 \Big(\partial_1 v(x_1, x_2) + h \partial_{y_1} v_1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}, x_3) \Big) \theta_0(\frac{x_1}{h}, \frac{x_2}{h}) \boldsymbol{e}_1$$
$$-h^4 \Big(\partial_2 v(x_1, x_2) + h \partial_{y_2} v_1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}, x_3) \Big) \theta_0(\frac{x_1}{h}, \frac{x_2}{h}) \boldsymbol{e}_2$$
$$-h^3 x_3^h \Big(\partial_1 v(x_1, x_2) \partial_1 \theta(\frac{x_1}{h}, \frac{x_2}{h}) + \partial_2 v(x_1, x_2) \partial_2 \theta(\frac{x_1}{h}, \frac{x_2}{h}) \Big) \boldsymbol{e}_3$$
$$+h^4 x_3^h \boldsymbol{d}^0(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}) + \frac{1}{2} h^2 (x_3^h)^2 \boldsymbol{d}^1(x_1, x_2, \frac{x_1}{h}, \frac{x_2}{h}), (4.78)$$

where $\boldsymbol{d}_0, \boldsymbol{d}_1 \in \mathcal{D}(\omega; C_\#^\infty(Y))^3$ are going to be chosen later. We calculate

$$\nabla \boldsymbol{y}^h \nabla \boldsymbol{\Theta}^h = \nabla \boldsymbol{\Theta}^h + h^2 \mathbf{A}' + h^4 \mathbf{B}' - h^2 x_3^h \mathbf{C}'$$
$$-h^3 \begin{pmatrix} \partial_1 v \partial_1 \theta & \partial_1 v \partial_2 \theta & 0 \\ \partial_2 v \partial_1 \theta & \partial_2 v \partial_2 \theta & 0 \\ 0 & 0 & \partial_1 v \partial_1 \theta + \partial_2 v \partial_2 \theta \end{pmatrix}$$
$$+h^4 \begin{pmatrix} -\partial_{y_1} v \partial_1 \theta & -\partial_{y_1} v \partial_2 \theta & 0 \\ -\partial_{y_2} v \partial_1 \theta & -\partial_{y_2} v \partial_2 \theta & 0 \\ \partial_{x_1} v_1 & \partial_{x_2} v_1 & 0 \end{pmatrix}$$
$$-h^2 x_3^h \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 v \partial_{11} \theta + \partial_2 v \partial_{12} \theta & \partial_1 v \partial_{12} \theta + \partial_2 v \partial_{22} \theta & 0 \end{pmatrix}$$
$$+h^4 \boldsymbol{d}_0 \otimes \boldsymbol{e}_3 + h^2 x_3^h \boldsymbol{d}_1 \otimes \boldsymbol{e}_3 + O(h^5), \qquad (4.79)$$

where $\|O(h^5)\|_{L^\infty(\Omega)} \leq Ch^5$ and

$$\mathbf{A}' = \begin{pmatrix} 0 & 0 & -\partial_1 v - h \partial_{y_1} v_1 \\ 0 & 0 & -\partial_2 v - h \partial_{y_2} v_1 \\ \partial_1 v + h \partial_{y_1} v_1 & \partial_2 v + h \partial_{y_2} v_1 & 0 \end{pmatrix}, \qquad (4.80)$$

$$\mathbf{B}' = \begin{pmatrix} \partial_1 \boldsymbol{u}^1 + \partial_{y_1} \boldsymbol{u}_1^1 & \partial_2 \boldsymbol{u}^1 + \partial_{y_2} \boldsymbol{u}_1^1 & 0 \\ \partial_1 \boldsymbol{u}^2 + \partial_{y_1} \boldsymbol{u}_1^2 & \partial_2 \boldsymbol{u}^2 + \partial_{y_2} \boldsymbol{u}_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} (\partial_{11} v + \partial_{y_1 y_1} v_1) \theta_0 & (\partial_{12} v + \partial_{y_1 y_2} v_1) \theta_0 & 0 \\ (\partial_{12} v + \partial_{y_1 y_2} v_1) \theta_0 & (\partial_{22} v + \partial_{y_2 y_2} v_1) \theta_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (4.81)$$

$$\mathbf{C}' = \begin{pmatrix} \partial_{11} v + \partial_{y_1 y_1} v_1 & \partial_{12} v + \partial_{y_1 y_2} v_1 & 0 \\ \partial_{12} v + \partial_{y_1 y_2} v_1 & \partial_{22} v + \partial_{y_2 y_2} v_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (4.82)$$

From (4.79), by using (2.4), we conclude

$$\nabla \boldsymbol{y}^h = \mathbf{I} + h^2 \mathbf{A}' + h^4 \mathbf{B}' - h^2 x_3^h \mathbf{C}'$$
$$+ h^4 \mathbf{O} + h^2 x_3^h \mathbf{P} + h^4 \boldsymbol{d}_0 \otimes \boldsymbol{e}_3 + h^2 x_3^h \boldsymbol{d}_1 \otimes \boldsymbol{e}_3 + O(h^5), \quad (4.83)$$

where

$$\mathbf{O} = \begin{pmatrix} 0 & 0 & o_{13}^h \\ 0 & 0 & o_{23}^h \\ o_{31}^h & o_{32}^h & o_{33}^h \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_{31}^h & p_{32}^h & 0 \end{pmatrix} \qquad (4.84)$$

$o_{i3}^h, o_{3i}^h, p_{3\alpha}^h \in \mathcal{D}(\omega; C_\#^\infty(Y))$, $\|o_{i3}^h\|_{L^\infty(\omega)}, \|o_{3i}^h\|_{L^\infty(\omega)}, \|p_{3\alpha}^h\|_{L^\infty(\omega)} \leq C$, C independent of h.

Let us define

$$\mathbf{R}^h := e^{h^2 \mathbf{A}}.$$

The claims (4.75)-(4.76) are easily checked to be valid as well as the convergence (4.11), (4.12).

Using the identities $(\mathbf{I} + \mathbf{A})^T (\mathbf{I} + \mathbf{A}) = \mathbf{I} + 2 \operatorname{sym} \mathbf{A} + \mathbf{A}^T \mathbf{A}$ and $(\boldsymbol{e}_3 \otimes \boldsymbol{a}' - \boldsymbol{a}' \otimes \boldsymbol{e}_3)^T (\boldsymbol{e}_3 \otimes \boldsymbol{a}' - \boldsymbol{a}' \otimes \boldsymbol{e}_3) = \boldsymbol{a}' \otimes \boldsymbol{a}' + |\boldsymbol{a}'|^2 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$ for $\boldsymbol{a}' \in \mathbb{R}^2$ we obtain

$$
\begin{aligned}
(\nabla \boldsymbol{y}^h)^T (\nabla \boldsymbol{y}^h) = \mathbf{I} &+ h^4 [2 \operatorname{sym} \mathbf{B} + \nabla v \otimes \nabla v + |\nabla v|^2 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3] \\
&- 2h^2 x_3^h (\nabla^2 v + \nabla_y^2 v_1) + 2h^4 \operatorname{sym}(\boldsymbol{d}_0 \otimes \boldsymbol{e}_3) + \\
&+ 2h^2 x_3^h \operatorname{sym}(\boldsymbol{d}_1 \otimes \boldsymbol{e}_3) + 2h^4 \operatorname{sym} \mathbf{O} + 2h^2 x_3^h \operatorname{sym} \mathbf{P} + O(h^5).
\end{aligned}
\tag{4.85}
$$

For a symmetric 2×2 matrix $\mathbf{A}''$ let $c = \mathcal{L} \mathbf{A}'' \in \mathbb{R}^3$ denote the (unique) vector which realizes the minimum in the definition of Q_2. i.e.

$$
Q_2(\mathbf{A}'') = Q_3(\mathbf{A}'' + \boldsymbol{c} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{c}).
$$

Since Q_3 is positive definite on symmetric matrices, $\boldsymbol{c}$ is uniquely determined and the map $\mathcal{L}$ is linear. We choose now

$$
\boldsymbol{d}_0^i = -\frac{1}{2} |\nabla v|^2 \delta_{i3} - \frac{1}{2} (\mathbf{O}_{i3} + \mathbf{O}_{3i}) + \mathcal{L}(\operatorname{sym} \mathbf{B} + \frac{1}{2} \nabla v \otimes \nabla v),
$$

$$
\boldsymbol{d}_1^i = -\frac{1}{2} (\mathbf{P}_{i3} + \mathbf{P}_{3i}) - \mathcal{L}(\nabla^2 v + \nabla_y^2 v_1),
$$

where δ is Kronecker symbol, and we can see that all the calculations are still valid. Taking the square root in (4.85) and using the frame indifference of W and the Taylor expansion we get

$$
\frac{1}{h^8} \int_{\hat{\Omega}^h} W(\nabla \boldsymbol{y}^h) = \frac{1}{h^8} \int_{\hat{\Omega}^h} W([(\nabla \boldsymbol{y}^h)^T \nabla \boldsymbol{y}^h]^{1/2}) \to I^L(\boldsymbol{u}, v, \boldsymbol{u}_1, v_1), \tag{4.86}
$$

In the case $\theta \in C_\#^\infty(Y)$ we could use Lemma (13). Since we supposed only $\theta \in C_\#^2(Y)$ we continue as follows. Let us now take an arbitrary $\boldsymbol{u} \in W^{1,2}(\omega)$, $v \in W^{2,2}(\omega)$. Let $\boldsymbol{u}_1(\nabla \boldsymbol{u}, -\nabla^2 v) \in L^2(\omega; \dot{H}_\#^1(Y))^2$, $v_1(\nabla \boldsymbol{u}, -\nabla^2 v) \in L^2(\omega; \dot{H}_\#^2(Y))$ be from Lemma 11. We know take $\boldsymbol{u}^n, v^n, \boldsymbol{u}_1^n, v_1^n$ smooth such that

$$
\|\boldsymbol{u}^n - \boldsymbol{u}\|_{W^{1,2}(\omega; \mathbb{R}^2)} \le \frac{1}{n} \tag{4.87}
$$

$$
\|v^n - v\|_{W^{2,2}(\omega)} \le \frac{1}{n} \tag{4.88}
$$

$$
\|\boldsymbol{u}_1^n - \boldsymbol{u}_1\|_{(L^2(\omega; \dot{H}_\#^1(Y))^2} \le \frac{1}{n} \tag{4.89}
$$

$$
\|v_1^n - v_1\|_{L^2(\omega; \dot{H}_\#^2(Y))} \le \frac{1}{n} \tag{4.90}
$$

$$
|I^L(\boldsymbol{u}^n, v^n, \boldsymbol{u}_1^n, v_1^n) - I_0^L(\boldsymbol{u}, v)| \le \frac{1}{n} \tag{4.91}
$$

Now, we choose $\boldsymbol{y}^{h_n} \in W^{1,2}(\hat{\Omega}^{h_n}; \mathbb{R}^3)$, from the proof, such that

i)

$$
\|(\nabla \boldsymbol{y}^{h_n}) \circ \boldsymbol{\Theta}^{h_n} \circ P^{h_n} - \mathbf{I}\|_{L^2(\Omega)} \le \frac{1}{n}
$$

ii) for v^{h_n} defined by (4.8) we have $\|v^{h_n} - v^n\|_{W^{1,2}(\omega)} \leq \frac{1}{n}$.

iii) for $\tilde{\boldsymbol{u}}^{h_n}$ defined by

$$\tilde{\boldsymbol{u}}^{h_n}(x_1, x_2) :=$$
$$\frac{1}{h_n^4}\left(\int_{-1/2}^{1/2} \begin{pmatrix} \boldsymbol{y}_1^{h_n} \circ \boldsymbol{\Theta}^{h_n} \circ P^{h_n} \\ \boldsymbol{y}_2^{h_n} \circ \boldsymbol{\Theta}^{h_n} \circ P^{h_n} \end{pmatrix}(x_1, x_2, x_3)dx_3 - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)$$
$$+ \begin{pmatrix} \partial_1 v^n(x_1, x_2)\theta_0(\frac{x_1}{h}, \frac{x_2}{h}) \\ \partial_2 v^n(x_1, x_2)\theta_0(\frac{x_1}{h}, \frac{x_2}{h}) \end{pmatrix},$$

we have $\|\tilde{\boldsymbol{u}}^{h_n} - \boldsymbol{u}^n\|_{L^2(\omega;\mathbb{R}^2)} \leq \frac{1}{n}$.

iv)

$$\|I^{h_n}(\boldsymbol{y}^{h_n}) - I^L(\boldsymbol{u}^n, v^n, \boldsymbol{u}_1^n, v_1^n)\| \leq \frac{1}{n}.$$

By using ii) and iii) and compactness Lemma 7 and Lemma 8 we conclude that there exists $\mathbf{R}^{h_n}$ such that (4.9), (4.10) is valid and for v^{h_n} and $\boldsymbol{u}^{h_n}$ defined by (4.8) we have that (4.11)-(4.17) is valid (it is easily seen from the properties ii) iii) and the assumptions (4.87) and (4.88) that the limits of $\boldsymbol{u}^{h_n}$ and v^{h_n} are $\boldsymbol{u}$ and v). By using (4.91) and iv) we conclude that

$$\lim_{n\to\infty} I^{h_n}(\boldsymbol{y}^{h_n}) = I_0^L(\boldsymbol{u}, v).$$

This finishes the proof of theorem.

Lemma 7, Lemma 10 and Theorem 7 enable us to standard theorem on convergence of minimizers. We shall state it without proof (since it is standard), assuming the external loads in $\boldsymbol{e}_3$ direction. For a more detailed discussion on external loads in the standard Föppl-von Kármán case see [17].

Let $\boldsymbol{f}_3^h \in L^2(\hat{\Omega}^h; \mathbb{R})$ be given with the property

$$\forall h, \frac{1}{h^6}\boldsymbol{f}_3^h \circ \boldsymbol{\Theta}^h \circ P^h = \boldsymbol{f}_3 \in L^2(\omega; \mathbb{R}), \tag{4.92}$$

$$\int_{\hat{\Omega}^h} \boldsymbol{f}_3^h dx = 0. \tag{4.93}$$

It is not necessary to prove the equality in (4.92) neither that $\boldsymbol{f}_3$ depends only on x_1, x_2. For a more detailed discussion see the proof of Theorem 2.5 in [18] (see also the proof of Theorem 6 in [24]). Let us additionally assume

$$\int_\omega x_1 \boldsymbol{f}_3 dx = 0, \quad \int_\omega x_2 \boldsymbol{f}_3 dx = 0. \tag{4.94}$$

This can be assumed by rotating the coordinate axis (the energy density W in rotational invariant). From (2.2) it can be concluded that

$$|\int_{\hat{\Omega}^h} x_1 \boldsymbol{f}_3^h dx| \leq Ch^4, \quad |\int_{\hat{\Omega}^h} x_2 \boldsymbol{f}_3^h dx| \leq Ch^4. \tag{4.95}$$

The total energy functional J^h (divided by h^2), defined on the space $W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$, is given by

$$J^h(\boldsymbol{y}^h) = h^8 I^h(\boldsymbol{y}^h) - \frac{1}{h^2} \int_{\hat{\Omega}^h} \boldsymbol{f}_3^h \boldsymbol{y}_3^h. \qquad (4.96)$$

The following theorem is the main result and its proof follows the proof of Theorem 2 in [12].

Theorem 8 (Γ-convergence) *Let us suppose that $\theta \in C_{\#}^2(Y)$ and $\boldsymbol{f}_3^h \in L^2(\hat{\Omega}^h; \mathbb{R})$ is given and satisfies (4.92), (4.93) and (4.94). Then:*

1. There exists $C_1, C_2 > 0$ such that for every $h > 0$ we have

$$C_1 h^2 \geq \inf \left\{ \frac{1}{h^8} J^h(\boldsymbol{y}^h); \boldsymbol{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3) \right\} \geq -C_2. \qquad (4.97)$$

In the case $\langle \theta \rangle \neq 0$ we can take $C_1 = 0$.

2. If $\boldsymbol{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ is a minimizing sequence of $\frac{1}{h^8} J^h$, that is

$$\lim_{h \to 0} \left(\frac{1}{h^8} J^h(\boldsymbol{y}^h) - \inf \frac{1}{h^8} J^h \right) = 0, \qquad (4.98)$$

then we have that there exists $\bar{\mathbf{R}}^h \in SO(3)$, $\boldsymbol{c}^h \in \mathbb{R}$ such that the sequence $(\bar{\mathbf{R}}^h, \boldsymbol{y}^h)$ has its subsequence (also not relabeled) with the following property:
  i) For the sequence $\widetilde{\boldsymbol{y}}^h := (\bar{\mathbf{R}}^h)^T \boldsymbol{y}^h - \boldsymbol{c}^h$ and $\boldsymbol{u}^h, v^h$ defined by (4.8) the following is valid

$$\boldsymbol{u}^h \rightharpoonup \boldsymbol{u} \quad \text{weakly in } W^{1,2}(\omega; \mathbb{R}^2),$$
$$v^h \to v \quad \text{in } W^{1,2}(\omega), \ v \in W^{2,2}(\omega).$$

Any accumulation point $(\boldsymbol{u}, v, \bar{\mathbf{R}})$ of the sequence $(\boldsymbol{u}^h, v^h, \bar{\mathbf{R}}^h)$ minimizes the functional

$$J_0^L(\boldsymbol{u}, v, \bar{\mathbf{R}}) = I_0^L(\boldsymbol{u}, v) - \bar{\mathbf{R}}_{33} \int_\omega \boldsymbol{f}_3(x_1, x_2)\big(v(x_1, x_2) + \theta(\frac{x_1}{h}, \frac{x_2}{h})\big) dx_1 dx_2, \qquad (4.99)$$

where I_0^L is defined in (4.60). Moreover, for $\boldsymbol{f}_3 \neq 0$ we have $\bar{\mathbf{R}}_{33} = 1$ or $\bar{\mathbf{R}}_{33} = -1$.

3. The minimum of the functional J_0^L exists in the space $W^{1,2}(\omega; \mathbb{R}^2) \times W^{2,2}(\omega) \times SO(3)$. If $\boldsymbol{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ is a minimizing sequence (not relabeled) of $\frac{1}{h^8} J^h$ then we have that

$$\lim_{h \to 0} \frac{1}{h^8} J^h(\boldsymbol{y}^h) = \min_{\boldsymbol{u} \in W^{1,2}(\omega; \mathbb{R}^2), \ v \in W^{2,2}(\omega), \ \bar{\mathbf{R}} \in SO(3)} J_0^L(\boldsymbol{u}, v, \bar{\mathbf{R}}). \qquad (4.100)$$

Remark 5 When we compare this model with the ordinary plate model of the Föppl-von Kármán type we see that in the energy expression we mix the term $\nabla^2 v$ which measures the bending of the plate with the term $\operatorname{sym} \nabla \boldsymbol{u} + \frac{1}{2} \nabla v \otimes \nabla v$ which measures the stretching of the plate. Thus the imperfect plate (periodically wrinkled) can cause this kind of behavior.

Remark 6 In the same way we could derive the linear model of periodically wrinkled plate, starting from $3D$ nonlinear theory (see [12]; see also [24]). The linear model would be like this model without the term $\nabla v \otimes \nabla v$ in the expression (4.60), where we defined I_0^L.

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